

# INVARIANTS OF HAMILTONIAN FLOW ON LOCALLY COMPLETE INTERSECTIONS

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**ABSTRACT.** We consider the Hamiltonian flow on complex complete intersection surfaces with isolated singularities, equipped with the Jacobian Poisson structure. More generally we consider complete intersections of arbitrary dimension equipped with Hamiltonian flow with respect to the natural top polyvector field, which one should view as a degenerate Calabi-Yau structure.

Our main result computes the coinvariants of functions under the Hamiltonian flow. In the surface case this is the zeroth Poisson homology, and our result generalizes those of Greuel, Alev and Lambre, and the authors in the quasihomogeneous and formal cases. Its dimension is the sum of the dimension of the top cohomology and the sum of the Milnor numbers of the singularities. In other words, this equals the dimension of the top cohomology of a smoothing of the variety.

More generally, we compute the derived coinvariants, which replaces the top cohomology by all of the cohomology. Still more generally we compute the  $\mathcal{D}$ -module which represents all invariants under Hamiltonian flow, which is a nontrivial extension (on both sides) of the intersection cohomology  $\mathcal{D}$ -module, which is maximal on the bottom but not on the top. For cones over smooth curves of genus  $g$ , the extension on the top is the holomorphic half of the maximal extension.

## 1. INTRODUCTION

We explain how to recover the top cohomology and Milnor numbers from complete intersection surfaces with isolated singularities via their Poisson structure. Namely, we prove that the zeroth Poisson homology of these surfaces is isomorphic to the direct sum of the top cohomology and vector spaces of dimension equal to the Milnor numbers of the singularities. We will generalize this result in two directions: to higher-dimensional complete intersections (replacing zeroth Poisson homology by coinvariants of Hamiltonian flow) and to all of the topological cohomology of the singular variety (by replacing coinvariants by derived coinvariants). We expose these results as a series of generalizations. Then, in §2, we explain and strengthen these results using  $\mathcal{D}$ -modules, which is also the key to their proof. In §3 we state our other main results, which are on the structure of the  $\mathcal{D}$ -module used in §2.

The proofs of all results in this and the next section will be postponed to §5, but see also §2.3 below for an outline of the proofs.

**1.1. Complete intersection surfaces with isolated singularities.** We work in the contexts of complex algebraic or complex analytic varieties (when we say “affine,” we mean closed subvarieties of  $\mathbf{A}^n$  for some  $n$ ). In this subsection we will restrict to the algebraic setting.

Let  $X$  be a surface which is an algebraic complete intersection in  $Y := \mathbf{A}^{k+2}$ . Then  $X$  is Poisson, equipped with the Jacobian Poisson structure, defined as follows. Suppose  $X$  is cut out by functions  $f_1, \dots, f_k$ , with  $\dim Y = k + 2$ . Let  $\Xi_Y := \partial_{x_1} \wedge \dots \wedge \partial_{x_{k+2}} \in \wedge^{k+2} T_Y$  be the top polyvector field on  $Y$ . Then we can define a bivector on  $Y$  by the formula

$$\pi := i_{\Xi_Y}(df_1 \wedge \dots \wedge df_k)$$

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The bivector  $\pi$  induces the following skew-symmetric bracket (satisfying the Leibniz rule):

$$\{g, h\} := i_\pi(\mathrm{d}g \wedge \mathrm{d}h) = i_{\Xi_Y}(\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k \wedge \mathrm{d}g \wedge \mathrm{d}h).$$

It is elementary, but important, to observe the following:

- (1)  $\pi$  is Poisson, i.e.,  $\{-, -\}$  satisfies the Jacobi identity;
- (2) The functions  $f_1, \dots, f_k$  are Poisson central (otherwise known as Casimirs), so that  $\pi$  and  $\{-, -\}$  descend to  $\mathcal{O}_X = \mathcal{O}_Y/(f_1, \dots, f_k)$ ;
- (3) The resulting Poisson structure is nondegenerate on the smooth locus, call it  $X^{\mathrm{smth}}$ , of  $X$ , i.e.,  $X^{\mathrm{smth}}$  is a (not necessarily affine) symplectic surface.

Briefly, the first fact holds because  $\{-, -\}$  has generic rank two; in fact the leaves of the Hamiltonian vector fields  $\xi_f := \{f, -\}$  are the level surfaces of  $f_1, \dots, f_k$ , and the restriction of  $\pi$  to each such level surface is Poisson (as is true for every bivector on a surface). The second fact follows immediately from the definition. The third fact follows because the smooth locus of  $X$  and the nondegeneracy locus of  $\pi$  are both the locus on  $X$  where  $\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k$  does not vanish.

We will restrict our attention to the case where  $X$  has only isolated singularities. Since  $X$  is a complete intersection (in particular, Cohen-Macaulay), it is therefore normal, i.e., global functions on  $X^{\mathrm{smth}}$  equal global functions on  $X$ . Thus, in this case,  $\pi$  is the *unique* Poisson structure which is nondegenerate on the smooth locus, up to scaling by invertible functions  $\mathcal{O}_X^\times$ . In particular, up to scaling,  $\pi$  is independent of the choice of embedding  $X \hookrightarrow Y$ . In other words, given an arbitrary normal surface, there might not exist such a  $\pi$ , but when it exists, it is unique up to scaling, and in the complete intersection case it always exists.

Our main result computes the zeroth Poisson homology of the Poisson algebra  $\mathcal{O}_X$ , i.e.,  $\mathrm{HP}_0(\mathcal{O}_X) := \mathcal{O}_X/\{\mathcal{O}_X, \mathcal{O}_X\}$ . Let  $H_{\mathrm{top}}^\bullet(X)$  denote the topological cohomology of  $X$  under the complex topology. Let  $X^{\mathrm{sing}} \subseteq X$  be the singular locus of  $X$ , which is finite by assumption. For every  $s \in X^{\mathrm{sing}}$ , let  $\mu_s$  be the Milnor number of the singularity at  $s \in X^{\mathrm{sing}}$ .

**Theorem 1.1.**  $\mathrm{HP}_0(\mathcal{O}_X) \cong H_{\mathrm{top}}^2(X) \oplus \bigoplus_{s \in X^{\mathrm{sing}}} \mathbf{C}^{\mu_s}$ .

Note that  $\mathrm{HP}_0(\mathcal{O}_X)$  also is independent (up to canonical isomorphism) of the choice of  $\pi$  up to scaling by invertible functions, and the Milnor numbers  $\mu_s$  are as well; thus, all objects in the theorem are intrinsic to  $X$ .

**Remark 1.2.** In the local case where  $X$  is either a formal or local analytic neighborhood of a singular point  $s$ , the RHS of the theorem reduces to  $\mathbf{C}^{\mu_s}$ , and the theorem was proved in [ES12a, Corollary 5.9] (in the formal setting, but the analytic case also follows from the material of [ES12a, §5.1, 5.2]). This relied primarily on [Gre75, Proposition 5.7.(iii)].

In the case when  $\mathcal{O}_X$  is a quasihomogeneous complete intersection with respect to some weights on  $\mathbf{A}^n$ , the RHS of Theorem 1.1 similarly reduces to  $\mathbf{C}^\mu$  for  $\mu = \mu_0$  the Milnor number of the origin. Again in this case, the theorem was proved in [ES12a, Theorem 5.21]. Moreover, one can describe the weight grading on  $\mathrm{HP}_0(\mathcal{O}_X)$ : it is isomorphic, as a graded vector space, to the singularity ring,  $\mathcal{O}_X/(\frac{\partial(f_1, \dots, f_k)}{\partial(x_{i_1}, \dots, x_{i_k})})$ , where  $\frac{\partial(f_1, \dots, f_k)}{\partial(x_{i_1}, \dots, x_{i_k})}$  is the determinant of the matrix of partial derivatives  $\frac{\partial f_p}{\partial x_{i_q}}$  for  $1 \leq p, q \leq k$ .

**Remark 1.3.** It was not essential above that  $Y = \mathbf{A}^{k+2}$ . Indeed, we could let  $Y$  be an arbitrary affine Calabi-Yau variety of dimension  $k+2$  and  $\Xi_Y \in \wedge^{k+2} T_Y$  a nonvanishing polyvector field (here and elsewhere, by Calabi-Yau, we mean only that there exists a nonvanishing global (algebraic) volume form, and do not require any compactness condition; cf. §1.3 below and its footnote). The construction of the Jacobian Poisson bivector, nondegenerate on  $X^{\mathrm{smth}}$ , in fact does not even require the affine or algebraic conditions (although our theorem does). More generally, we could let  $Y$  be any (not necessarily affine) smooth analytic variety, and take the complete intersection of sections

$f_1 \in \mathcal{L}_1, \dots, f_k \in \mathcal{L}_k$  of line bundles  $\mathcal{L}_1, \dots, \mathcal{L}_k$  whose tensor product is isomorphic to  $\wedge^{\dim Y} T_Y$ . Still more generally, we could let  $f_1, \dots, f_k$  be sections of a vector bundle  $\mathcal{V}$  of rank  $k$  whose top exterior power is isomorphic to  $\wedge^{\dim Y} T_Y$ . Then  $\pi$  still is constructed as above (one needs to pick a connection locally, but the result is independent of the choice of connection), and in the case  $X$  is affine, the theorem applies with the same proof.

**1.1.1. Restatement in terms of the smoothing of  $X$ .** Since  $X$  is a complete intersection, it is equipped with a smoothing,  $\pi : \mathcal{X} \rightarrow \mathbf{A}^1$ , so that  $X = \pi^{-1}(0)$  and  $X_t := \pi^{-1}(t)$  is smooth for generic  $t$ . Let us denote the Betti numbers by  $h^i(Z) := \dim H_{\text{top}}^i(Z)$  for any topological space  $Z$ . Then it is well-known that  $h^2(X_t) = h^2(X) + \sum_s \mu_s$  for generic  $t$ : this is a consequence of the fact [Mil68, Ham71] that, for every  $s \in X^{\text{sing}}$  and  $0 < |t| \ll 1$ , the intersection of  $X_t$  with a small ball about  $s \in \mathcal{X}$  is homotopic to a bouquet of  $\mu_s$  2-spheres. We conclude that, for generic  $t$ ,  $\text{HP}_0(\mathcal{O}_{X_t}) \cong \text{HP}_0(\mathcal{O}_X)$ . In other words, the theorem is equivalent to the following result.

**Corollary 1.4.** The sheaf  $\text{HP}_0(X_t)$  on the line  $\mathbf{A}^1$  is a vector bundle near  $t = 0$  of rank  $h^2(X) + \sum_{s \in X^{\text{sing}}} \mu_s$ . The generic fiber is  $H_{\text{top}}^2(X_t)$ .

Note that being a vector bundle is the same as having fibers of constant dimension.

**1.2. Generalization to locally complete intersections.** More generally, we can let  $X$  be an arbitrary affine surface with isolated singularities at  $X^{\text{sing}} \subseteq X$  which is, near each  $s \in X^{\text{sing}}$ , Zariski locally an algebraic complete intersection (in an affine space). Still more generally, we could assume only that  $X$  is analytically locally an analytic complete intersection (in a polydisc).

Moreover we assume that  $X$  is equipped with a Poisson structure which vanishes only at the singular locus. Then we again prove Theorem 1.1, stated in exactly the same way. As before, when this Poisson structure exists, it is unique up to multiplication by a nonvanishing function. Moreover the choice of such Poisson structure does not affect the statement of the theorem, as remarked earlier. So the Poisson structure is a condition on  $X$ , not a structure.

Moreover, provided a smoothing  $X_t$  exists, Corollary 1.4 follows. Note that we do not need a global deformation over  $\mathbf{A}^1$ ; we could work with smoothing over a formal disc  $\text{Spf } \mathbf{C}[[t]]$ .

**Remark 1.5.** Our results generalize to the case where  $X$  need not admit a Poisson structure nonvanishing on  $X^{\text{smth}}$ , but admits a flat connection on  $T_X^2 := \text{Hom}(\wedge^2 \Omega_X^1, \mathcal{O}_X)$ , considered in [ES12a, §3.5] (we continue to require that  $X$  is analytically locally a complete intersection in a polydisc and has isolated singularities). For example, when  $X$  is itself smooth, then in the analytic setting this says that the universal cover of  $X$  has trivial canonical bundle and admits a flat connection invariant under deck transformations.

The reason why a flat connection on  $T_X^2$  suffices is because our arguments only require the notion of Hamiltonian vector fields, not that of Poisson bivectors. Indeed, if  $H(X)$  is the Lie algebra of Hamiltonian vector fields, then  $\text{HP}_0(\mathcal{O}_X) = (\mathcal{O}_X)_{H(X)}$ , the coinvariants of  $\mathcal{O}_X$  under  $H(X)$ .

Now, given a flat connection  $\nabla : T_X^2 \rightarrow T_X^2 \otimes \Omega_X^1$ , for every local section  $\eta \in \Gamma(U, T_X^2)$ , we can define the Hamiltonian vector field on  $U$ ,  $\xi_{\eta, f} := i_\eta(df) + f \text{ctr}(\nabla(\eta))$ , where if  $\nabla(\eta) = \sum_j \theta_j \otimes \alpha_j$ , then  $\text{ctr}(\nabla(\eta)) = \sum_j i_{\theta_j}(\alpha_j)$ . Using this, we define a presheaf of Hamiltonian vector fields, and we can take its sheafification. Then global sections of the latter form a Lie algebra, call it  $LH(X)$ , and we can consider  $(\mathcal{O}_X)_{LH(X)}$ .

With this definition,  $\xi_{g\eta, f} = \xi_{\eta, fg}$ . Therefore, if  $\eta \in \Gamma(U, T_X^2)$  is nonvanishing, all Hamiltonian vector fields are of the form  $\xi_{g\eta, f} = \xi_{\eta, fg}$  for some  $f, g \in \mathcal{O}_X$ . Moreover, if  $\eta$  is a flat section of  $\nabla$  on  $U \subseteq X$ , then  $\xi_{\eta, f} = i_\eta(df)$ . Therefore, when there is locally a flat nonvanishing section  $\pi$  of  $T_X^2$ , then Hamiltonian vector fields are locally the same as those associated to the Poisson bivector  $\pi$ .

Conversely, if  $X$  is Poisson (and nondegenerate on  $X^{\text{smth}}$ ), then  $T_X^2$  has a flat connection uniquely determined such that the Poisson bivector is a flat section (since  $X$  has only isolated singularities).

Then the Lie algebra  $LH(X)$  defined above is the Lie algebra of vector fields which are locally Hamiltonian with respect to  $\pi$ .

Note that, for  $X$  Poisson (and nondegenerate on  $X^{\text{smth}}$ ), the Lie algebra  $LH(X)$  may be larger than  $H(X)$ : elements of  $LH(X)$  are of the form  $\eta_\alpha := i_\pi \alpha$  where  $\alpha$  is only locally exact (whereas  $H(X)$  requires  $\alpha$  to be globally exact, so that there exists a Hamiltonian function  $f$  with  $\eta_\alpha = \eta_{df} = \xi_f$ ). However all of the arguments of this paper, which are of a local nature, are independent of this distinction, so under the hypotheses of this subsection,  $(\mathcal{O}_X)_{LH(X)} = (\mathcal{O}_X)_{H(X)} = \text{HP}_0(\mathcal{O}_X)$ , and therefore the main results of this paper all extend to the setting of this remark.

**1.3. Generalization to higher dimensional varieties.** We generalize this result to the case where  $\dim X \geq 2$ , replacing  $\text{HP}_0(\mathcal{O}_X)$  by the coinvariants  $(\mathcal{O}_X)_{H(X)}$  of functions  $\mathcal{O}_X$  under the Lie algebra of Hamiltonian vector fields defined in [ES12a, §3.4], which we recall below.

Briefly, when  $X \subseteq Y := \mathbf{A}^{n+k}$  is a complete intersection of dimension  $n$  by functions  $f_1, \dots, f_k$ , then  $X$  is equipped with a canonical polyvector field,

$$\Xi_X := i_{\Xi_Y}(\mathbf{d}f_1 \wedge \dots \wedge \mathbf{d}f_k)|_X.$$

Namely, it is easy to check that  $i_{\Xi_Y}(\mathbf{d}f_1 \wedge \dots \wedge \mathbf{d}f_k) \in \wedge^{\dim X} T_Y$  is parallel to  $X$  and thus restricts to a top polyvector field  $\Xi_X$  on  $X$ . Moreover, as before,  $\Xi_X$  is nondegenerate on  $X^{\text{smth}}$ , i.e.,  $(X^{\text{smth}}, \Xi_X^{-1})$  is Calabi-Yau. Here and henceforth, by Calabi-Yau we only mean a smooth variety together with a nonvanishing global volume form (we do not require any compactness condition).<sup>1</sup>

**Remark 1.6.** As in Remark 1.3, we could more generally take  $Y$  Calabi-Yau of dimension  $n+k$  with  $\Xi_Y \in \wedge^{n+k} T_Y$  nonvanishing, or an arbitrary smooth  $Y$  and the complete intersection of sections of a vector bundle whose top exterior power is  $\wedge^{\dim Y} T_Y$ .

In the case that  $X$  has isolated singularities (in fact, even if it has singularities of codimension at most two), then once again  $\Xi_X$  is the *unique* top polyvector field on  $X$  which is nonvanishing on  $X^{\text{smth}}$ , up to scaling by  $\mathcal{O}_X^\times$ .

As observed in [ES12a, §3.4], one can define Hamiltonian vector fields on  $(X, \Xi_X)$  by contracting  $\Xi_X$  with exact  $(\dim X - 1)$ -forms. Then Theorem 1.1 generalizes to

**Theorem 1.7.** For affine  $X$  as above,  $(\mathcal{O}_X)_{H(X)} \cong H_{\text{top}}^{\dim X}(X) \oplus \bigoplus_{s \in X^{\text{sing}}} \mathbf{C}^{\mu_s}$ .

More generally, as in the surface case, we prove this theorem when  $X$  is an arbitrary affine locally complete intersection of dimension  $\geq 2$  with isolated singularities equipped with a polyvector field  $\Xi_X$  which is nondegenerate on the smooth locus, i.e., such that  $X^{\text{smth}}$  is Calabi-Yau. Here as before “locally complete intersection” means (in the analytic context) that every singular point  $s \in X^{\text{sing}}$  has an analytic neighborhood which is an analytic complete intersection in a polydisc. When  $X$  is algebraic, this includes the case where  $s$  has a Zariski neighborhood which is a complete intersection. We remark, as before, that since  $\Xi_X$  is unique up to multiplication by a nonvanishing function, it is clear that  $(\mathcal{O}_X)_{H(X)}$  does not depend on the choice of  $\Xi_X$  up to isomorphism.

**Remark 1.8.** As in Remark 1.2, in the case that  $X$  is a formal or local analytic neighborhood of a singular point  $s$ , the theorem was proved in [ES12a, Corollary 5.9], using [Gre75, Proposition 5.7.(iii)]. Similarly, when  $X$  is conical, i.e., it is affine and admits a contracting  $\mathbf{C}^\times$  action, and  $\Xi_X$  is assumed to be homogeneous of some weight, the result is a straightforward generalization of

<sup>1</sup>We remark that every smooth variety is locally Calabi-Yau in our sense, e.g., by taking any local top differential form and restricting to its nonvanishing locus. This is probably why in some places the Calabi-Yau condition is accompanied by a compactness condition. This also explains the perhaps initially surprising fact that, for instance, smooth hypersurfaces in affine space of arbitrary degree are Calabi-Yau in our sense (as opposed to the case of projective space, where only hypersurfaces of degree one more than the dimension of the projective space can be Calabi-Yau); also affine space itself is Calabi-Yau in our sense. Viewed differently, our definition is a higher-dimensional generalization of symplectic surfaces, where no compactness condition is generally imposed.

[ES12a, Theorem 5.21], where the graded vector space structure is proved to identify with that of the singularity ring.

Parallel to Corollary 1.4, we have the result

**Corollary 1.9.** Suppose that  $X_t$  is a smoothing of  $X$ , i.e.,  $\pi : \mathcal{X} \rightarrow \mathbf{A}^1$  is a family with  $X_t := \pi^{-1}(t)$  generically smooth and  $X = X_0$ . Then, the sheaf  $(\mathcal{O}_{X_t})_{H(X_t)}$  on  $\mathbf{A}^1$  is a vector bundle near  $t = 0$  of rank  $h^{\dim X}(X) + \sum_{s \in X^{\text{sing}}} \mu_s$ . The generic fiber is the top cohomology  $H_{\text{top}}^{\dim X}(X_t)$ .

Provided a smoothing exists, this is equivalent to the theorem. However, since  $X$  is only locally a complete intersection, we only know that there exists a smoothing of a neighborhood of each singular point. Note that we could alternatively work with a smoothing over a formal disc  $\text{Spf } \mathbf{C}[[t]]$ , and the corollary extends to this case.

**Remark 1.10.** Parallel to Remark 1.5, all of our main results generalize to the case where  $X$  is not necessarily equipped with a top polyvector field  $\Xi_X$  nonvanishing on  $X^{\text{smth}}$ , but is instead equipped with a flat connection  $\nabla$  on  $T_X^{\dim X} := \text{Hom}(\wedge^{\dim X} \Omega_X^1, \mathcal{O}_X)$ , cf. [ES12a, §3.5], in addition to being analytically locally a complete intersection in a polydisc and having isolated singularities. This is enough to define, as before, the Lie algebra  $LH(X)$  of vector fields which are locally Hamiltonian on  $X$ , and the aforementioned results are all valid replacing  $(\mathcal{O}_X)_{H(X)}$  by  $(\mathcal{O}_X)_{LH(X)}$ . In the case where  $\nabla$  actually admits a flat section  $\Xi_X$ , then as in Remark 1.5,  $H(X)$  could be a proper Lie subalgebra of  $LH(X)$ , the latter which is the contraction of  $\Xi_X$  with locally exact (rather than exact)  $(n-1)$ -forms. Nonetheless, as before, one has  $(\mathcal{O}_X)_{LH(X)} \cong (\mathcal{O}_X)_{H(X)}$ .

**1.4. A question on symmetric powers.** In [ES12b], in the case that  $X$  is a conical hypersurface in  $\mathbf{C}^3$  (so that  $X^{\text{sing}} = \{0\}$  is the vertex), we computed the zeroth Poisson homology of arbitrary symmetric powers of  $X$ .

Let  $\lambda \vdash n$  denote that  $\lambda$  is a partition of  $n$ , which we write as  $\lambda = (\lambda_1, \dots, \lambda_m)$  for  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m$  and  $\lambda_1 + \dots + \lambda_m = n$ . In this case, define  $|\lambda| := m$ . For each such partition, let  $S_\lambda < S_m$  denote the group of permutations preserving the partition. Explicitly,  $S_\lambda = S_{r_1} \times \dots \times S_{r_k}$  where, for all  $j$ ,

$$\lambda_{r_1+\dots+r_j} > \lambda_{r_1+\dots+r_j+1} = \lambda_{r_1+\dots+r_j+2} = \dots = \lambda_{r_1+\dots+r_j+r_{j+1}}.$$

Define also, for an arbitrary vector space  $V$ ,

$$S^\lambda V := (V^{\otimes |\lambda|})^{S_\lambda}.$$

In [ES12b], we proved that, for  $X$  a conical symplectic surface in  $\mathbf{C}^3$  with an isolated singularity,

$$(1.11) \quad \text{HP}_0(\mathcal{O}_{S^n X}) \cong \bigoplus_{\lambda \vdash n} S^\lambda \text{HP}_0(\mathcal{O}_X),$$

and, by [AL98],  $\text{HP}_0(\mathcal{O}_X) \cong \mathbf{C}^{\mu_s}$  in this case. By definition,  $\text{HP}_0(\mathcal{O}_{S^n X}) = (\mathcal{O}_{S^n X})_{H(S^n X)}$ . As observed in [ES12b, Proposition 1.4.1], this also equals  $(\mathcal{O}_{S^n X})_{H(X)}$ , where  $H(X)$  acts on  $S^n X$  by the diagonal action. This allows us to state a generalization of the above equality when  $X$  is an arbitrary complete intersection (and not just for surfaces):

**Question 1.12.** Does the analogue of (1.11) hold when  $X$  is replaced by an arbitrary locally complete intersection with isolated singularities? That is, does one have

$$(1.13) \quad (\mathcal{O}_{S^n X})_{H(X)} \cong \bigoplus_{\lambda \vdash n} S^\lambda ((\mathcal{O}_X)_{H(X)})?$$

By Theorem 1.7, (1.13) can be rewritten as

$$(1.14) \quad (\mathcal{O}_{S^n X})_{H(X)} \cong \bigoplus_{\lambda \vdash n} S^\lambda (H_{\text{top}}^{\dim X}(X) \oplus \bigoplus_{s \in X^{\text{sing}}} \mathbf{C}^{\mu_s}).$$

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## 2. DERIVED COINVARIANTS, ALL DE RHAM COHOMOLOGY, AND $\mathcal{D}$ -MODULES

**2.1.  $\mathcal{D}$ -module generalization for surfaces.** In order to prove the above results we actually prove a much more general result, which also computes the *derived coinvariants* of  $\mathcal{O}_X$  under Hamiltonian flow. To formulate these, we use  $\mathcal{D}$ -modules. These also have the advantage of being local, so that the final statements we prove for complete intersections will immediately imply the generalization to locally complete intersections.

**2.1.1. The  $\mathcal{D}$ -module  $M(X)$ .** We need to recall the essential construction of [ES10]. Let  $X$  be an affine Poisson (complex algebraic or complex analytic) variety and  $H(X)$  its Lie algebra of Hamiltonian vector fields. Let  $M(X)$  be the right  $\mathcal{D}$ -module defined in [ES10] representing invariants under Hamiltonian flow. We briefly recall its definition:

$$M(X) := H(X) \cdot \mathcal{D}_X \setminus \mathcal{D}_X,$$

where  $\mathcal{D}_X$  is the right  $\mathcal{D}$ -module on  $X$  representing global sections, i.e.,  $\Gamma(X, M) = \text{Hom}(\mathcal{D}_X, M)$  for all right  $\mathcal{D}$ -modules; this carries a canonical left action by vector fields on  $X$ , and  $H(X) \cdot \mathcal{D}_X$  is then a right  $\mathcal{D}$ -submodule of  $\mathcal{D}_X$ .

Then, by definition, one has [ES10, Proposition 2.13]:

$$(2.1) \quad \text{HP}_0(\mathcal{O}_X) \cong \pi_0 M(X),$$

where  $\pi : X \rightarrow \text{Spec } \mathbf{C}$  is the projection to point and  $\pi_0 M(X) = H_0(\pi_* M(X))$  is the underived pushforward.

Recall the definition of higher Poisson-de Rham homology (for an arbitrary Poisson variety  $X$ ):

**Definition 2.2.** [ES10, Remark 2.27]  $\text{HP}_*^{DR}(X) := \pi_* M(X)$ .

(As explained in [ES10, Remark 2.27], for  $X$  symplectic, this coincides with the de Rham cohomology of  $X$ :  $\text{HP}_*^{DR}(X) \cong H_{\text{top}}^{\dim X - *}(X)$ .)

Moreover, as explained in [ES10], all of the above generalizes to arbitrary (not necessarily affine) algebraic or analytic varieties  $X$ , when one replaces  $H(X)$  by the *presheaf*  $\mathcal{H}(X)$  of Hamiltonian vector fields. (We remark that  $\mathcal{H}(X)$  is *not* necessarily a sheaf: see Remark 1.5 above and [ES12a, Remark 4.5]; this does not cause any trouble in the definition of  $M(X)$ , as explained there.)

**2.1.2. Poisson-de Rham homology of (locally) complete intersection surfaces.** Now let  $X$  be a surface as in §1.2. As before, this includes the cases of an algebraic complete intersection in affine space or in a Calabi-Yau variety (or the more general example of Remark 1.3). We do not need  $X$  to be affine.

Then we have the following result which generalizes Theorem 1.1:

**Theorem 2.3.**  $\text{HP}_i^{DR}(X) \cong \begin{cases} H_{\text{top}}^{2-i}(X), & \text{if } i > 0, \\ H_{\text{top}}^2 \oplus \bigoplus_{s \in X^{\text{sing}}} \mathbf{C}^{\mu_s}, & \text{if } i = 0. \end{cases}$

Restricting to the case where  $X$  is a complete intersection in affine space there always exists a smoothing  $X_t$  of  $X$ . Then Theorem 2.3 is equivalent to the following analogue of Corollary 1.4. As before,  $h^i(X) := \dim H^i(X)$  denotes the  $i$ -th Betti number of  $X$ .

**Corollary 2.4.** The sheaf of graded vector spaces  $\mathrm{HP}_*^{DR}(X_t)$  is a vector bundle near  $t = 0$ . Its generic fiber is  $H_{\mathrm{top}}^{\dim X - *}(X_t)$ , and the Hilbert series of the fibers near  $t = 0$  is  $h(\mathrm{HP}_*^{DR}(X_t); u) = \sum_m h^{\dim X - m}(X) u^m + \sum_{s \in X^{\mathrm{sing}}} \mu_s$ .

Note that being a graded vector bundle is merely saying that the Hilbert series of the fibers is constant.

The same corollary holds in the general case (so  $X$  is only analytically locally a complete intersection, and not necessarily affine), provided it is equipped with a smoothing  $X_t$  (as before one could also work with a smoothing over a formal disc).

**2.1.3. The  $\mathcal{D}$ -module of (locally) complete intersection surfaces.** Let  $X$  be as in the previous subsection. Let  $j : X^{\mathrm{smth}} \hookrightarrow X$  be the inclusion of the smooth locus into  $X$ . Let  $\Omega_{X^{\mathrm{smth}}}$  be the canonical right  $\mathcal{D}$ -module of volume forms on  $X^{\mathrm{smth}}$ . Then we can consider the underived push-forward  $H^0 j_! \Omega_{X^{\mathrm{smth}}}$ . This is the maximal indecomposable extension on the bottom (by  $\mathcal{D}$ -modules supported on  $X^{\mathrm{sing}}$ ) of the intersection cohomology  $\mathcal{D}$ -module  $\mathrm{IC}(X) := j_{!*} \Omega_{X^{\mathrm{smth}}}$ , where  $j_{!*}$  is the intermediate extension. More precisely, we have the following standard result, for which we provide a proof for the reader's convenience:

**Proposition 2.5.**  $N := H^0 j_! \Omega_{X^{\mathrm{smth}}}$  is indecomposable, and is the universal extension of the form

$$(2.6) \quad 0 \rightarrow K \rightarrow N \rightarrow \mathrm{IC}(X) \rightarrow 0,$$

with  $K$  supported on  $X^{\mathrm{sing}}$ , in the sense that every other extension

$$(2.7) \quad 0 \rightarrow K' \rightarrow N' \rightarrow \mathrm{IC}(X) \rightarrow 0,$$

with  $K'$  supported on  $X^{\mathrm{sing}}$ , is the pushout of a canonical morphism  $K \rightarrow K'$ .

*Proof.* First of all, by adjunction, we have a canonical morphism in the derived category,  $j_! \Omega_{X^{\mathrm{smth}}} \rightarrow \mathrm{IC}(X)$ , which becomes the identity after applying  $j^!$ . Since  $j_! \Omega_{X^{\mathrm{smth}}}$  is a complex concentrated in nonpositive (cohomological) degrees, it follows that, for all  $\mathcal{D}$ -modules  $L$ ,

$$(2.8) \quad H^0 \mathrm{RHom}(j_! \Omega_{X^{\mathrm{smth}}}, L) \cong \mathrm{Hom}(H^0 j_! \Omega_{X^{\mathrm{smth}}}, L) = \mathrm{Hom}(N, L).$$

In more detail, take the exact triangle  $j_! \Omega_{X^{\mathrm{smth}}} \rightarrow H^0 j_! \Omega_{X^{\mathrm{smth}}} \rightarrow C$ , for  $C = \mathrm{cone}(j_! \Omega_{X^{\mathrm{smth}}} \rightarrow H^0 j_! \Omega_{X^{\mathrm{smth}}})$ . The cohomology of  $C$  is concentrated in negative degrees. Then the long exact sequence for  $\mathrm{Hom}(-, L)$  yields, for all  $\mathcal{D}$ -modules  $L$ , the exact sequence  $0 \rightarrow \mathrm{Hom}(H^0 j_! \Omega_{X^{\mathrm{smth}}}, L) \rightarrow H^0 \mathrm{RHom}(j_! \Omega_{X^{\mathrm{smth}}}, L) \rightarrow 0$ , since  $H^i(C) = 0$  for  $i \geq 0$ .

Thus, by (2.8), we obtain from adjunction that

$$(2.9) \quad \mathrm{Hom}(N, \mathrm{IC}(X)) = H^0 \mathrm{RHom}(j_! \Omega_{X^{\mathrm{smth}}}, \mathrm{IC}(X)) \cong \mathrm{Hom}(\Omega_{X^{\mathrm{smth}}}, \Omega_{X^{\mathrm{smth}}}) = \mathbf{C}.$$

We therefore obtain the canonical extension (2.6).

To see that  $N$  is indecomposable, note that, by the same computation of (2.9),  $\mathrm{Hom}(N, N) \cong H^0 \mathrm{RHom}(j_! \Omega_{X^{\mathrm{smth}}}, N) \cong \mathrm{Hom}(\Omega_{X^{\mathrm{smth}}}, \Omega_{X^{\mathrm{smth}}}) = \mathbf{C}$ .

Finally, we show that  $N$  is the universal (maximal indecomposable) extension of  $\mathrm{IC}(X)$  by  $\mathcal{D}$ -modules supported on  $X^{\mathrm{sing}}$ . If not, let  $N'$  be this extension (it exists since  $X^{\mathrm{sing}}$  is finite and  $\mathrm{Ext}(\mathrm{IC}(X), \delta_s)$  is finite-dimensional for all  $s \in X^{\mathrm{sing}}$ ). By universality, the surjection  $N' \rightarrow \mathrm{IC}(X)$  factors through  $N \rightarrow \mathrm{IC}(X)$ , but by adjunction and (2.8),  $\mathrm{Hom}(N, N') = \mathbf{C}$ . Thus we obtain a nonzero composition  $N \rightarrow N' \rightarrow N$ , which must be a nonzero multiple of the identity since  $\mathrm{Hom}(N, N) = \mathbf{C}$ . Since  $N'$  is indecomposable, this implies that  $N = N'$ , a contradiction.  $\square$

We can make (2.6) more explicit as follows. For every  $s \in X^{\mathrm{sing}}$ , let  $\delta_s$  denote the  $\delta$ -function  $\mathcal{D}$ -module at  $s$ . Then,

$$(2.10) \quad K \cong \bigoplus_{s \in X^{\mathrm{sing}}} \mathrm{Ext}^1(\mathrm{IC}(X), \delta_s)^* \otimes \delta_s.$$

This can be written in terms of topological cohomology: for every  $s \in X^{\text{sing}}$ , let  $U_s$  be a contractible neighborhood of  $s$  (whose existence was proved in [Gil64], cf. also [Mil68, 2.10]), disjoint from  $X^{\text{sing}} \setminus \{s\}$ . Then by a straightforward generalization of [ES10, Lemma 4.3],

$$(2.11) \quad \text{Ext}^1(\text{IC}(X), \delta_s) \cong H^1(U_s \setminus \{s\}).$$

Thus (2.10) can be alternatively written as

$$(2.12) \quad K \cong \bigoplus_{s \in X^{\text{sing}}} H^1(U_s \setminus \{s\}) \otimes \delta_s.$$

Then Theorems 2.3 and 1.1 follow from the following result.

**Theorem 2.13.** There is an exact sequence

$$0 \rightarrow H^0 j_! \Omega_{X^{\text{smth}}} \rightarrow M(X) \twoheadrightarrow \bigoplus_{s \in X^{\text{sing}}} \mathbf{C}^{\mu_s} \otimes \delta_s \rightarrow 0.$$

One advantage of using  $\mathcal{D}$ -modules is that this is a local statement: so the statement above for complete intersections now immediately implies the generalization to the case where  $X$  is only analytically locally complete intersection as in §1.2 (and not necessarily affine). Indeed, we only need to observe that the  $\mathcal{D}$ -module  $M(X)$  does not depend on the choice of  $\Xi_X$ , up to isomorphism, since  $\Xi_X$  is unique up to multiplication by an everywhere nonvanishing function. In more detail, if  $f \in \Gamma(X, \mathcal{O}_X)$  is everywhere nonvanishing, one has the isomorphism  $M(X, \Xi_X) \rightarrow M(X, f\Xi_X)$ , which on affine open subsets  $U \subseteq X$  sends the canonical generator  $1 \in M(U, \Xi_X|_U)$  to  $f^{-1}$  times the canonical generator of  $M(U, f\Xi_X|_U)$ .

We can also prove an analogue of Corollaries 1.4 and 2.4 (which we will actually prove simultaneously with the preceding theorem). Suppose that  $X_t$  is a smoothing of  $X$  (over  $\mathbf{A}^1$  or over a formal disc  $\text{Spf } \mathbf{C}[[t]]$ ); this always exists if  $X$  is a complete intersection. Let  $\mathcal{X}$  be the total space of the smoothing.

**Theorem 2.14.** The sheaf over  $\mathbf{A}^1$  of (fiberwise)  $\mathcal{D}$ -modules  $M(X_t)$  on  $\mathcal{X}$  is flat near  $t = 0 \in \mathbf{A}^1$ . For generic  $t \in \mathbf{A}^1$ , the fiber at  $t$  is  $\Omega_{X_t}$ .

In the general case where  $X$  is only locally a complete intersection, we can apply the theorem to the open complex neighborhoods  $U_s$  of each  $s \in X^{\text{sing}}$  which are analytically complete intersections. This is an important ingredient in the proof of Theorem 2.13 itself.

**2.2.  $\mathcal{D}$ -modules on higher dimensional varieties.** Theorems 2.3 and 2.13 also carry over verbatim to the higher-dimensional setting with  $X$  a “degenerate Calabi-Yau” analytically locally complete intersection with finite degeneracy locus. Precisely, we let  $X$  be a not necessarily affine analytically locally complete intersection of dimension  $\geq 2$  equipped with a top polyvector field  $\Xi_X$  vanishing at only finitely many points, i.e.,  $X$  has only isolated singularities and  $\Xi_X$  vanishes only at the singular locus.

Then, we prove Theorem 2.13 in this general context (stated in exactly the same way), and this implies all of the preceding theorems. Note that, in this case, the maximal extension  $H^0 j_! \Omega_{X^{\text{smth}}}$  of  $\text{IC}(X)$  (by  $\mathcal{D}$ -modules supported on  $X^{\text{sing}}$ ) is an extension by

$$(2.15) \quad K \cong \bigoplus_{s \in X^{\text{sing}}} \text{Ext}^1(\text{IC}(X), \delta_s)^* \otimes \delta_s \cong \bigoplus_{s \in X^{\text{sing}}} H^{\dim X - 1}(U_s \setminus \{s\}) \otimes \delta_s,$$

where  $U_s$  is a contractible neighborhood of  $s$  for every  $s \in X^{\text{sing}}$ , disjoint from  $X^{\text{sing}} \setminus \{s\}$ .

The analogue of Theorem 2.3, which follows from the above, is as follows. Let  $\pi : X \rightarrow \text{Spec } \mathbf{C}$  be the projection to a point.



**Corollary 2.16.**  $\pi_i M(X) \cong \begin{cases} H_{\text{top}}^{\dim X - i}(X), & \text{if } i > 0, \\ H_{\text{top}}^{\dim X}(X) \oplus \bigoplus_{s \in X^{\text{sing}}} \mathbf{C}^{\mu_s}, & \text{if } i = 0. \end{cases}$

Equivalently, if  $X_t$  is a smoothing of  $X$  (which exists at least when  $X$  is a global complete intersection; as before we could work with a smoothing over a formal disc or over  $\mathbf{A}^1$ ), this becomes the following (with  $h^i(X) = \dim H^i(X)$  again the  $i$ -th Betti number of  $X$ ):

**Corollary 2.17.** The sheaf of graded vector spaces  $\pi_* M(X_t)$  on  $\mathbf{A}^1$  is a graded vector bundle near  $t = 0$  whose fibers have Hilbert series  $h(\pi_* M(X_t), u) = \sum_m h^{\dim X - m}(X) u^m + \sum_{s \in X^{\text{sing}}} \mu_s$ . The generic fiber is  $H_{\text{top}}^{\dim X - *}(X_t)$ .

Note that being a graded vector bundle is the same as saying that the Hilbert series of the fibers is constant.

Finally, we also prove Theorem 2.14 in this context, which goes through verbatim.

**2.3. Outline of the proof.** We outline the proof of Theorems 2.13 and 2.14 (which imply all of the results from §1 and §2) in the original situation of  $X$  a complete intersection surface in affine space. For details, see §5.

- (1) We exploit the smoothing  $X_t$ , where generically  $X_t$  is smooth and hence  $M(X_t) = \Omega_{X_t}$  (by [ES10, Example 2.6], since then  $X_t$  is a symplectic surface).
- (2) We exploit the structure theory of  $M(X)$ : it must be an extension of  $IC(X)$  on (possibly) both sides by delta-function  $\mathcal{D}$ -modules at  $X^{\text{sing}}$ . We use that the maximal extension  $H^0 j_! \Omega_{X^{\text{smth}}}$  is by  $K$  as in (2.12).
- (3) By [ES12a, §5], the maximal quotient supported at  $X^{\text{sing}}$  is  $\bigoplus_{s \in X^{\text{sing}}} \text{HP}_0(\hat{\mathcal{O}}_{X,s}) \otimes \delta_s$ , and  $\dim \text{HP}_0(\hat{\mathcal{O}}_{X,s}) = \mu_s$ .
- (4) Now,  $M(X_t)$  is flat near  $t = 0$  if and only if there is no torsion at  $t = 0$ ; this torsion would have to be supported at  $X^{\text{sing}}$ .
- (5) We take the Euler-Poincaré characteristic of  $\pi_* M(X_t)$  (for all  $t$ ). Since the torsion of the family  $M(X_t)$  is a direct sum of delta-function  $\mathcal{D}$ -modules, the Euler-Poincaré characteristic of  $\pi_* M(X_t)$  can only increase at  $t = 0$ , and it increases if and only if the family is not flat (i.e.,  $\pi_* M(X_t)$  is not flat viewed as a coherent sheaf on  $\mathbf{A}^1$ ).
- (6) Using the classical formula [GM80, §6.1] for  $\pi_* IC(\bar{X})$  for compact  $\bar{X}$  (applied to the one-point compactification of a contractible neighborhood of each isolated singularity), we compute the Euler-Poincaré characteristic of  $\pi_* IC(X)$  (Proposition 5.9).
- (7) This formula shows that the Euler-Poincaré characteristic of  $\pi_* M(X_t)$  cannot increase at  $t = 0$ , so the family is flat, proving Theorem 2.14. The formula simultaneously shows that  $M(X)$  maximally extends  $IC(X)$  on the bottom (otherwise the Euler-Poincaré characteristic of  $\pi_* M(X_t)$  would go down at  $t = 0$ , which is impossible), which proves Theorem 2.13.

The exact same outline above applies to the situation where  $X$  is a complete intersection of arbitrary dimension, except that  $X_t$ , rather than being a symplectic surface, is a smooth Calabi-Yau variety for generic  $t$ , so  $M(X_t) \cong \Omega_{X_t}$  still holds by [ES12a, Example 2.37]. The proof is then the same, but we have to use in step (6) the fact that, for an analytically locally complete intersection of dimension  $n$  with an isolated singularity, the link of the singularity is  $(n - 2)$ -connected [Mil68], [Ham71, Korollar 1.3].

Once we have the results for complete intersections, as observed, we immediately obtain them for locally complete intersections, since Theorems 2.13 and 2.14 are local. In fact, we prove the theorem in the local setting, assuming  $X$  is contractible with a single isolated singularity, which implies the global result.

### 3. THE FULL STRUCTURE OF $M(X)$ AND TOP LOG DIFFERENTIAL FORMS ON RESOLUTIONS

As before we assume that  $X$  is an analytically locally complete intersection equipped with a top polyvector field  $\Xi_X$  which is nondegenerate on  $X^{\text{smth}}$  and  $\dim X \geq 2$ . Theorem 2.13 implies that  $M(X)$  is a direct sum of delta-function  $\mathcal{D}$ -modules together with one indecomposable extension by  $H^0 j_! \Omega_{X^{\text{smth}}}$  of delta-function  $\mathcal{D}$ -modules supported on  $X^{\text{sing}}$ . Let  $M_{\max}(X)$  denote this indecomposable extension.

To fully describe the structure of  $M(X)$  and  $M_{\max}(X)$ , in view of the exact sequence of Theorem 2.13, we only need to describe how much of the quotient  $\bigoplus_s \mathbf{C}^{\mu_s} \otimes \delta_s$  therein splits off of  $M(X)$ , and how much is in the image of  $M_{\max}(X)$ . That is, it remains only to compute the quotient  $M_{\max}(X)/H^0 j_! \Omega_{X^{\text{smth}}}$ , i.e., the maximal quotient of  $M_{\max}(X)$  supported on  $X^{\text{sing}}$ .

In this section, in Theorem 3.2 and Proposition 3.4, we do this when  $X$  is conical, or more generally locally conical near all the singular points. For the general case, we state Conjecture 3.11, which we show follows from the aforementioned results in the locally conical case in Proposition 3.15. The proofs of Theorem 3.2 and Proposition 3.4 are postponed to §7.

Here and below, a *conical* variety  $X$  means an affine variety with a contracting  $\mathbf{C}^\times$  action to a fixed point. In the algebraic setting this means that  $\mathcal{O}_X$  is nonnegatively graded (by what we will call the *weight* grading) with  $\mathbf{C}$  in weight zero; in the analytic setting  $\mathcal{O}_X$  is a completion of the direct sum of its homogeneous components, which are in nonnegative weights and with  $\mathbf{C}$  in weight zero. We will also use the term “weight” in general to refer to the grading induced by a  $\mathbf{C}^\times$  action.

**3.1. The case of a  $\mathbf{C}^\times$ -action.** Suppose that  $X$  is equipped with a  $\mathbf{C}^\times$  action, e.g.,  $X$  is conical (and hence affine). Moreover, assume that  $\Xi_X$  is homogeneous of some weight  $d$ . Then, the symplectic volume form on  $X^{\text{smth}}$  is homogeneous of weight  $-d$ . Next,  $j^* M(X) = M_{X^{\text{smth}}} \cong \Omega_{X^{\text{smth}}}$ , and the symplectic volume form is a distributional solution of  $j^* M(X)$ , so for every affine open subset  $U \subseteq X^{\text{smth}}$ , the canonical generator  $1 \in M_U$  is annihilated by the operator  $\text{Eu} - d$  (as the right action of  $\text{Eu}$  on a distribution  $\phi$  of weight  $m$  is  $\phi \cdot \text{Eu} = -m\phi$ ; alternatively, on volume forms  $\text{Eu}$  acts by  $-L_{\text{Eu}}$ , which gives the same answer).

Now,  $M(X^{\text{smth}})$  is weakly equivariant with respect to the  $\mathbf{C}^\times$  action. Since the canonical generator on every affine open has weight  $d$ , this implies that  $M(X^{\text{smth}})$  is homogeneous of weight  $d$  as a weakly  $\mathbf{C}^\times$ -equivariant  $\mathcal{D}$ -module (the difference of the two canonical actions of the Lie algebra  $\mathbf{C} \cdot \text{Eu}$  of  $\mathbf{C}^\times$  is the character  $\text{Eu} \mapsto d$ ). Therefore,  $H^0 j_! M(X^{\text{smth}}) \cong H^0 j_! \Omega_{X^{\text{smth}}}$  is also weakly equivariant of weight  $d$ , and this is the structure such that the canonical morphism  $H^0 j_! M(X^{\text{smth}}) \rightarrow M(X)$  is a morphism of weakly equivariant  $\mathcal{D}$ -modules.

Decompose  $M(X)$  as  $M(X) = \bigoplus_{m \in M} M(X)_m$ , with  $M(X)_m$  the part of weight  $m$ . The morphism  $H^0 j_! M(X^{\text{smth}}) \rightarrow M(X)$  lands entirely in weight  $d$ . Also,  $j^* M(X)_m = 0$  for  $m \neq d$  (which also follows directly because the canonical morphism  $j^* M(X)_m \rightarrow M(X^{\text{smth}})_m = 0$  is zero). So, for  $m \neq d$ ,  $M(X)_m$  is a direct sum of delta function  $\mathcal{D}$ -modules on points of the finite set  $X^{\text{sing}}$ , whereas at  $m = d$ , the extension of Theorem 2.13 splits as follows:

$$(3.1) \quad 0 \rightarrow H^0 j_! \Omega_{X^{\text{smth}}} \rightarrow M(X)_d \rightarrow \bigoplus_{s \in X^{\text{sing}}} E_s \rightarrow 0,$$

where  $E_s$  is the weight  $d$  component of  $\mathbf{C}^{\mu_s} \otimes \delta_s$  in Theorem 2.13.

**Theorem 3.2.** Suppose that  $s \in X^{\text{sing}}$  is an attracting or repelling fixed point of the  $\mathbf{C}^\times$ -action. Let  $Y := X^{\text{smth}} \cup \{s\}$ , and  $j_Y : Y^{\text{smth}} = X^{\text{smth}} \hookrightarrow Y$  be the inclusion. Then, the extension

$$(3.3) \quad 0 \rightarrow H^0(j_Y)_! \Omega_{Y^{\text{smth}}} \rightarrow M(Y)_d \rightarrow E_s \rightarrow 0$$

is indecomposable. In particular, if all of the points  $s \in X^{\text{sing}}$  are attracting or repelling fixed points (e.g.,  $X$  is conical with  $X^{\text{sing}} = \{s\}$ ), then the extension (3.1) is indecomposable.

The theorem will be proved in the case where  $X$  is a homogeneous hypersurface in  $\mathbf{C}^3$  in §6 below (and hence also where a neighborhood of  $s$  is a neighborhood of such a hypersurface). The proof will then be adapted to the general case in §7.

We can also explicitly describe  $E_s$ . In a neighborhood  $U_s$  of  $s \in X^{\text{sing}}$ , let  $U_s$  be a complete intersection in a polydisc, and consider the formal completion  $\hat{\mathcal{O}}_{U_s, s} = \hat{\mathcal{O}}_{X, s}$  (or one could use the analytic or algebraic local ring). Then  $\hat{\mathcal{O}}_{X, s}$  also has a  $\mathbf{C}^\times$  action and decomposes into weight spaces,  $\hat{\mathcal{O}}_{X, s} = \prod_m (\hat{\mathcal{O}}_{X, s})_m$ .

**Proposition 3.4.** Suppose  $s$  is an attracting or repelling fixed point. Then, there is a canonical isomorphism  $E_s \cong (\hat{\mathcal{O}}_{X, s})_d \otimes \delta_s$ .

This proposition will also be proved in §§6 and 7 below (it follows relatively easily from results we will recall in §5.3).

**Example 3.5.** If  $X$  is the cone over a smooth curve of weight  $d$  in the projective plane  $\mathbf{CP}^2$ , then one may check that  $\dim(\mathcal{O}_X)_d = g$ , where  $g = \frac{(d-1)(d-2)}{2}$  is the genus of the curve. In fact, for arbitrary conical surfaces  $X$ , if  $X^{\text{smth}}/\mathbf{C}^\times$  is a curve of genus  $g$ , then it is still true that  $\dim(\mathcal{O}_X)_d = g$ : this follows from Example 3.14 and Propositions 3.13 and 3.15 below. Since, in this case,  $H^0 j_! \Omega_{X^{\text{smth}}}$  is an extension of the irreducible  $\mathcal{D}$ -module  $\text{IC}(X)$  by  $\delta_s^{2g}$  (by (2.11)), we see that  $M(X)_{\max}$  has a filtration with subquotients  $\delta_s^g$  on the top,  $\text{IC}(X)$  in the middle, and  $\delta_s^{2g}$  on the bottom. That is, the extension is maximal on the bottom, and half-maximal on the top.

**Remark 3.6.** The fact that (3.3) is indecomposable is a local statement, i.e., it can be applied to neighborhoods  $U_s$  of each point  $s$ . Thus, we deduce the following statement: If  $s \in X$  has a neighborhood  $U_s$  which is isomorphic to a neighborhood  $V_s$  of a conical variety  $Y$ , then  $M(U_s)_{\max} \cong M(Y)|_{V_s}$  is given by an exact sequence, for  $U_s^\circ := U_s \setminus \{s\}$ ,

$$(3.7) \quad 0 \rightarrow H^0(j_{U_s^\circ})_! \Omega_{U_s} \rightarrow M(U_s)_{\max} \rightarrow (\hat{\mathcal{O}}_{Y, s})_d \otimes \delta_s \rightarrow 0.$$

**Remark 3.8.** Note that, in the previous remark, we did not need the assumption that the top polyvector field  $\Xi_{V_s}$  on  $U_s \cong V_s$  was the restriction of a homogeneous polyvector field on  $Y$ . Assume only that  $\Xi_{V_s}$  is nonvanishing on  $V_s^\circ$ . We claim that there exists a homogeneous polyvector field  $\Xi_Y$  on  $Y$  which is nonvanishing on  $Y^\circ := Y \setminus \{s\}$ . Since, as we remarked before,  $H(V_s)$  does not depend on the choice of  $\Xi_{V_s}$  nonvanishing on  $V_s^\circ$  (since  $H(V_s)$  does not change when multiplying  $\Xi_{V_s}$  by any nonvanishing function),  $M(V_s)$  is independent of this choice as well. Thus we can assume without loss of generality that  $\Xi_{V_s}$  is the restriction of a homogeneous polyvector field  $\Xi_Y$  on  $Y$ .

To prove the claim, we first restrict to a formal (or analytic) completion  $\hat{X}_s$ . Write  $\Xi_{\hat{X}_s} := \Xi_{V_s}|_{\hat{X}_s}$  as a sum of homogeneous components,  $\Xi_{\hat{X}_s} = \sum_{m=N}^\infty (\Xi_{\hat{X}_s})_m$ . Here  $N \in \mathbf{Z}$  exists because, for some  $N \in \mathbf{Z}$ , one has  $(T_{\hat{X}_s}^{\dim X})_{<N} = 0$ : for instance, if  $\omega \in \Omega_Y^{\dim X}$  is any homogeneous top differential form, of weight  $-N \geq 0$ , then  $(T_{\hat{X}_s}^{\dim X})(\omega) \subseteq \hat{\mathcal{O}}_{\hat{X}_s}$ , and hence  $(T_{\hat{X}_s}^{\dim X})_{<N} = 0$ .

Now, it cannot be that, for all  $m$ ,  $(\Xi_{\hat{X}_s})_m \in \mathfrak{m}_s \cdot \Xi_{\hat{X}_s}$ , for  $\mathfrak{m}_s$  the maximal ideal of  $s$ , since in that case  $\Xi_{\hat{X}_s} \in \mathfrak{m}_s \cdot \Xi_{\hat{X}_s}$ , which is impossible. For all  $m$ , write  $(\Xi_{\hat{X}_s})_m = f_m \cdot \Xi_{\hat{X}_s}$ . Then for some  $m$ ,  $f_m$  is invertible in a neighborhood of  $s$  (and we remark that this implies that  $f_k = 0$  for  $k < m$ , since  $f_k f_m^{-1}$  is regular on  $Y$  of weight  $k - m$ ). Therefore,  $f_m^{-1} \Xi_{\hat{X}_s} = (\Xi_{\hat{X}_s})_m$  is homogeneous. Now, homogeneous elements of  $T_{\hat{X}_s}$  are regular on all of  $Y$ , since  $Y$  is conical. Thus,  $(\Xi_{\hat{X}_s})_m$  is the restriction of a regular homogeneous top polyvector field  $\Xi_Y$  on  $Y$  which is nonvanishing on  $Y^\circ$ , as desired.

**3.2. General conjecture on the indecomposable summand of  $M(X)$ .** For any complex algebraic or analytic variety  $X$ , let  $\Omega_X^\bullet$  denote the commutative differential graded algebra of

Kähler forms on  $X$ . Recall that this is the algebra generated by  $\mathcal{O}_X$  in degree zero and symbols  $\mathbf{d}f$ , for all  $f \in \mathcal{O}_X$ , in degree one, subject to the relations

$$\mathbf{d}(fg) = \mathbf{d}(f) \cdot g + f \cdot \mathbf{d}(g), \quad f, g \in \mathcal{O}_X,$$

and equipped with the differential, which we also denote abusively by  $\mathbf{d}$ , which is the unique derivation sending  $f$  to  $\mathbf{d}f$  for all  $f \in \mathcal{O}_X$  and annihilating the symbols  $\mathbf{d}f$  for all  $f \in \mathcal{O}_X$ .

When  $X$  is not necessarily an affine variety, then we let  $\Omega_X^\bullet$  denote the quasicoherent sheaf of Kähler forms on  $X$ . Note that the commutative differential graded algebra structure is compatible with restriction, i.e., this is a sheaf of differential graded  $\mathcal{O}_X$ -algebras.

For every  $s \in X^{\text{sing}}$ , let  $U_s$  be a neighborhood of  $s$  which is a complete intersection in a polydisc (which is disjoint from  $X^{\text{sing}} \setminus \{s\}$ ). Let  $U_s^\circ := U_s \setminus \{s\}$  be the punctured neighborhood.

**Definition 3.9.** Let  $\Omega_{\text{conv}}^{\dim X}(U_s^\circ) \subseteq \Omega^{\dim X}(U_s^\circ)$  denote the vector space of holomorphic  $(\dim X)$ -forms  $\alpha$  on  $U_s^\circ$  which are  $L^2$ -convergent at  $s$ , i.e., such that  $\int \alpha \wedge \bar{\alpha}$  is convergent in a neighborhood of  $s$ .

**Definition 3.10.** Let  $\Omega_{\text{log}}^{\dim X}(U_s^\circ) \subseteq \Omega^{\dim X}(U_s^\circ)$  denote the vector space of holomorphic  $(\dim X)$ -forms  $\alpha$  on  $U_s^\circ$  which are at most logarithmically divergent at  $s$ . That is, for every function  $f \in \mathcal{O}_{U_s}$  vanishing at  $s$ , the limit  $|\log \varepsilon|^{-1} \int_{|f| \geq \varepsilon} \alpha \wedge \bar{\alpha}$  exists as  $\varepsilon \rightarrow 0$ .

We can give an alternative description of the above spaces: Let  $\widetilde{U}_s \rightarrow U_s$  be a resolution of singularities with a normal crossing exceptional divisor  $D$ . Then,  $\Omega_{\text{conv}}^{\dim X}(U_s^\circ)$  consists of those forms which extend to holomorphic forms on  $\widetilde{U}_s$ , and  $\Omega_{\text{log}}^{\dim X}(U_s^\circ)$  consists of those forms which extend to meromorphic forms on  $\widetilde{U}_s$  with at most first-order poles (on  $D$ ).

**Conjecture 3.11.**  $M_{\text{max}}$  is an extension by  $H^0 j_! \Omega_{X^{\text{smth}}}$  of

$$(3.12) \quad \bigoplus_s \Omega_{\text{log}}^{\dim X}(U_s^\circ) / \Omega_{\text{conv}}^{\dim X}(U_s^\circ) \otimes \delta_s.$$

We can give an alternative description of the space in the conjecture, in terms of an arbitrary resolution  $\widetilde{U}_s \rightarrow U_s$  with normal crossing exceptional divisor  $D$ :

**Proposition 3.13.** The vector space  $\Omega_{\text{log}}^{\dim X}(U_s^\circ) / \Omega_{\text{conv}}^{\dim X}(U_s^\circ)$  is canonically isomorphic to  $\Gamma(D, \Omega^{\dim D}(D))$ .

In other words, this is the direct sum of the vector spaces of holomorphic volume forms on the components of  $D$ .

*Proof.* Consider the exact sequence of sheaves on  $\widetilde{U}_s$ ,

$$0 \rightarrow \Omega_{\widetilde{U}_s}^{\dim X} \rightarrow \Omega_{\widetilde{U}_s}^{\dim X}(D) \rightarrow \Omega_D^{\dim X-1} \rightarrow 0,$$

where  $\Omega_{\widetilde{U}_s}^{\dim X}(D)$  is the space of meromorphic  $(\dim X)$ -forms on  $\widetilde{U}_s$  with at most simple poles on  $D$  (and no poles elsewhere), and the second nontrivial map takes the residue along the components of  $D$ . The long exact sequence on cohomology yields, in view of the comments before the conjecture,

$$0 \rightarrow \Omega_{\text{conv}}^{\dim X}(U_s^\circ) \rightarrow \Omega_{\text{log}}^{\dim X}(U_s^\circ) \rightarrow \Omega_D^{\dim X-1} \rightarrow H^1(\widetilde{U}_s, \Omega_{\widetilde{U}_s}^{\dim X}).$$

The proposition will follow once we show that  $H^1(\widetilde{U}_s, \Omega_{\widetilde{U}_s}^{\dim X})$  is zero. But, this is a consequence of the Grauert-Riemenschneider vanishing theorem, since  $\rho : \widetilde{U}_s \rightarrow U_s$  is a resolution of singularities of an affine variety. In more detail, the Grauert-Riemenschneider vanishing theorem for proper birational  $\rho$  implies that the complex  $\rho_* \Omega_{\widetilde{U}_s}^{\dim X}$  has vanishing cohomology sheaves in nonzero degrees.

Thus  $R^i \rho_* \Omega_{\widetilde{U}_s}^{\dim X} = 0$  for  $i > 0$ . Therefore  $H^1(\widetilde{U}_s, \Omega_{\widetilde{U}_s}^{\dim X}) = H^1(U_s, \rho_0 \Omega_{\widetilde{U}_s}^{\dim X})$ , where  $\rho_0 \Omega_{\widetilde{U}_s}^{\dim X}$  is

the underived pushforward of  $\Omega_{\widetilde{U}_s}^{\dim X}$ , i.e., it is a sheaf (rather than a complex of sheaves), or alternatively a complex of sheaves with cohomology concentrated in degree zero. Since  $U_s$  is affine, the higher cohomology of any sheaf on  $U_s$  is zero. Thus,  $H^1(\widetilde{U}_s, \Omega_{\widetilde{U}_s}^{\dim X}) = 0$ .  $\square$

**Example 3.14.** If  $X$  is a surface, then  $D$  is a curve. Therefore, for  $D_i$  the components of  $D$ ,

$$\Gamma(D, \Omega_D^1) \cong \bigoplus_{D_i} \Gamma(D_i, \Omega_{D_i}^1),$$

and hence the dimension of  $\Gamma(D, \Omega_D^1)$  is the sum of the genera of the components  $D_i$ . Since  $\widetilde{U}_s \rightarrow U_s$  was arbitrary, this sum must be independent of the choice of the resolution  $\widetilde{U}_s$ , and this fact is well-known.

**3.3. Proof of the conjecture in the locally conical case.** In this subsection we prove the following result using §3.1. Call  $X$  *locally conical* at  $s$  if there exists a neighborhood  $U_s$  of  $s$  which is isomorphic to a neighborhood of a conical variety. Equivalently,  $(U_s, s) \cong (V_s, s)$  where  $V_s \subseteq Y$  is an open subset and  $Y$  is conical with vertex  $s$ .

**Proposition 3.15.** Conjecture 3.11 is true when, for every  $s \in X^{\text{sing}}$ ,  $X$  is locally conical at  $s$ .

*Proof.* Since Conjecture 3.11 is a local statement, we can assume  $X$  is conical with vertex  $s$ . Thus  $X$  is affine and equipped with a  $\mathbf{C}^\times$  action with  $s$  as attracting fixed point. As explained in Remark 3.8, we can further assume that  $\Xi_{U_s}$  is the restriction of a homogeneous polyvector field on  $X$  of some weight  $d$ , which is therefore nonvanishing on  $X^\circ := X \setminus \{0\}$ . For simplicity, we first work on  $X$  instead of on  $U_s$ .

Thus, by Theorem 3.2 and Remark 3.6,  $M_{\max}$  is an extension by  $H^0 j! \Omega_{X^{\text{smth}}}$  of  $(\mathbf{C}^{\mu_s} \otimes \delta_s)_d$ , where the latter is the weight  $d$  part of the maximal quotient of  $M(X)$  supported at  $s$ . By Proposition 3.4, the latter is isomorphic to  $(\hat{\mathcal{O}}_{X,s})_d \otimes \delta_s$ .

Next, note that  $\Xi_X^{-1}$  is a holomorphic  $(\dim X)$ -form on  $X^\circ$ , and all holomorphic  $(\dim X)$ -forms on  $X^\circ$  are of the form  $\mathcal{O}_X \cdot \Xi_X^{-1}$ .

Note that a holomorphic  $(\dim X)$ -form on  $X$  automatically has positive weight (at least  $\dim X$ ), since  $X$  is conical. Furthermore, a homogeneous holomorphic  $(\dim X)$ -form on  $X^\circ$  is at most logarithmically divergent if and only if it has nonnegative weight; similarly, it is convergent if and only if it has positive weight. Therefore,

$$\Omega_{\log}^{\dim X}(X^\circ) = (\mathcal{O}_X)_{\geq d} \cdot \Xi_X^{-1}, \quad \Omega_{\text{conv}}^{\dim X}(X^\circ) = (\mathcal{O}_X)_{\geq (d+1)} \cdot \Xi_X^{-1}.$$

We may conclude that  $\Omega_{\log}^{\dim X}(X^\circ)/\Omega_{\text{conv}}^{\dim X}(X^\circ) \cong (\mathcal{O}_X)_d$ . Since  $X$  is conical, the latter is the same as  $(\hat{\mathcal{O}}_{X,s})_d$ .

Now, if we work instead on  $U_s$ , we can use a formal (or analytic) completion  $\hat{X}_s$  at  $s$ . Then, the same argument as above shows that

$$\Omega_{\log}^{\dim X}(U_s^\circ) = ((\hat{\mathcal{O}}_{X,s})_{\geq d} \cap \mathcal{O}_{U_s}) \cdot \Xi_{U_s}^{-1}, \quad \Omega_{\text{conv}}^{\dim X}(U_s^\circ) = ((\hat{\mathcal{O}}_{X,s})_{\geq (d+1)} \cap \mathcal{O}_{U_s}) \cdot \Xi_{U_s}^{-1}.$$

Consider the canonical inclusions

$$(3.16) \quad ((\hat{\mathcal{O}}_{X,s})_{\geq d} \cap \mathcal{O}_X) / ((\hat{\mathcal{O}}_{X,s})_{\geq (d+1)} \cap \mathcal{O}_X) \rightarrow ((\hat{\mathcal{O}}_{X,s})_{\geq d} \cap \mathcal{O}_{U_s}) / ((\hat{\mathcal{O}}_{X,s})_{\geq (d+1)} \cap \mathcal{O}_{U_s}) \\ \rightarrow (\hat{\mathcal{O}}_{X,s})_{\geq d} / (\hat{\mathcal{O}}_{X,s})_{\geq (d+1)} \cong (\mathcal{O}_X)_d,$$

whose composition is an isomorphism since  $X$  is conical. Hence, the middle term is also isomorphic to  $(\mathcal{O}_X)_d$ , so  $\Omega_{\log}^{\dim X}(U_s^\circ)/\Omega_{\text{conv}}^{\dim X}(U_s^\circ) \cong (\mathcal{O}_X)_d \cong (\hat{\mathcal{O}}_{X,s})_d$  as well.  $\square$

#### 4. RECOLLECTIONS ON HAMILTONIAN VECTOR FIELDS

As before, throughout the paper  $X$  can be either a complex algebraic or complex analytic variety, and  $\Omega_X^\bullet$  is its differential graded algebra of Kähler forms.

Here we recall from [ES12a, §3.4] the definition of the Lie algebra of Hamiltonian vector fields  $H(X)$  when  $X$  is equipped with a top polyvector field  $\Xi_X \in \wedge^{\dim X} T_X$ . In the case when  $X$  is a complete intersection surface in affine space (or in a Calabi-Yau variety, or as in Remark 1.3), this is the Hamiltonian flow with respect to the Jacobian Poisson structure on  $X$ . More generally, for complete intersections in Calabi-Yau varieties of higher dimensions, we have the corresponding top polyvector field.

Associated to  $\Xi_X$ , as in [ES12a, §3.4], is a natural Lie algebra of “Hamiltonian” vector fields. Namely, to every  $(n-2)$ -form  $\alpha \in \Omega_X^{n-2}$ , we associate the Hamiltonian vector field  $\xi_\alpha := i_{d\alpha} \Xi_X$ .

**Definition 4.1.** [ES12a, §3.4] Let  $H(X) := \{\xi_\alpha \mid \alpha \in \Omega_X^{n-2}\}$ .

**Proposition 4.2.** [ES12a, §3.4]  $H(X)$  is a Lie subalgebra of the Lie algebra of vector fields under the Lie bracket.

Essentially, the preceding proposition is proved by observing the identity

$$[\xi_\alpha, \xi_\beta] = \xi_{i_{\xi_\alpha} d\beta},$$

which implies that  $[H(X), H(X)] \subseteq H(X)$ .

**4.1. Global generalization.** More generally, if  $Y$  or  $X$  is not necessarily affine, we can make the same definition as in Definition 4.1, where now  $H(X)$  should be replaced by the *presheaf*  $\mathcal{H}(X)$  of Hamiltonian vector fields. Then, Proposition 4.2 carries over to show that  $\mathcal{H}(X)$  is a presheaf of Lie algebras, i.e.,  $[\mathcal{H}(X), \mathcal{H}(X)] = \mathcal{H}(X)$ . See [ES12a, §2.10, Corollary 4.4] for details.

#### 5. PROOF OF RESULTS FROM §1 AND §2

As pointed out in §2, all of the theorems in §1 are implied by the more general versions in §2, and in particular, everything follows from Theorems 2.13 and 2.14, in their more general form where  $X$  is allowed to be analytically locally complete intersections  $X$  of arbitrary dimension  $\geq 2$  equipped with a top polyvector field  $\Xi_X$  which vanishes at only finitely many points, i.e., at  $X^{\text{sing}}$ . To prove these results, we follow the steps in the outline of the proof in §2.3, with subsections here numbered the same as the outline there.

As the statements are local, it suffices to replace  $X$  with an open neighborhood of each  $s \in X^{\text{sing}}$ , i.e., to assume  $|X^{\text{sing}}| = 1$ . Note that if  $X$  is smooth, then the theorems immediately follow from the fact that, in this case,  $M(X) \cong \Omega_{X^{\text{smth}}}$  by [ES12a, Example 2.6]).

It suffices to assume that  $X$  is an analytic complete intersection in a polydisc  $Y$ . We henceforth assume this. Let  $\dim X = n$  and say that  $X$  is cut out of  $Y \cong B^{n+k}$  by  $k$  analytic functions,  $f_1, \dots, f_k \in \mathcal{O}_Y$ . Up to shrinking  $Y$ , we can assume in fact that  $X$  is contractible ([Gil64], cf. also [Mil68, 2.10]).

**5.1. The smoothing  $X_t$ .** For generic  $c_1, \dots, c_k$ , it follows that the complete intersection cut out by  $f_1 - c_1, \dots, f_k - c_k$  is smooth. Therefore we can pick  $c_1, \dots, c_k$  such that the one-parameter family  $X_t := Z(f_1 - tc_1, \dots, f_k - tc_k) \subseteq Y$  is generically smooth. As mentioned in the introduction, by [Mil68, Ham71], for a small enough open ball  $B(s)$  about  $s$  and for  $0 < |t| \ll 1$ ,  $X_t$  is smooth and the intersection  $X_t \cap B(s)$  is homotopic to a bouquet of  $\mu_s$  spheres of dimension  $n$ . Let us shrink  $Y$  so that  $X_t$  is itself homotopic to a bouquet of  $\mu_s$  spheres of dimension  $n$  for  $t \in D \subseteq \mathbf{C}$ , where  $D$  is a suitably small open ball about 0; we also assume that  $X_t$  is smooth for  $t \in D \setminus \{0\}$ .

**5.2. The structure of the  $\mathcal{D}$ -module  $M(X)$ .** As explained in [ES12a, Example 2.37],  $M(X^{\text{smth}}) \cong \Omega_{X^{\text{smth}}}$ , since  $X^{\text{smth}}$  is Calabi-Yau. As in the introduction, let  $j : X^{\text{smth}} \hookrightarrow X$  be the inclusion, so that  $M(X^{\text{smth}}) = j^! M(X) = j^* M(X)$ , which is therefore  $\Omega_{X^{\text{smth}}}$ . Then by adjunction we have a canonical morphism  $N := H^0 j_! \Omega_{X^{\text{smth}}} = H^0 j_! j^! M(X) \rightarrow M(X)$ . This is an isomorphism on  $X^{\text{smth}}$ , and hence the cokernel, call it  $K'$ , is supported on  $X^{\text{sing}}$ . Thus we have the exact sequence

$$(5.1) \quad N \rightarrow M(X) \rightarrow K' \rightarrow 0.$$

Theorem 2.13 reduces to showing that the first morphism,  $N \rightarrow M(X)$ , is injective. We will show this in §5.7 below.

As observed in §2.1.3 and §2.2,  $N$  is an indecomposable extension of the simple  $\mathcal{D}$ -module  $\text{IC}(X) = j_{!*} \Omega_{X^{\text{smth}}}$ , of the form

$$(5.2) \quad 0 \rightarrow K \rightarrow N \rightarrow \text{IC}(X) \rightarrow 0, \quad K = \text{Ext}^1(\text{IC}(X), \delta_s)^* \otimes \delta_s \cong H^{n-1}(X \setminus \{s\}) \otimes \delta_s.$$

For the final equality, we used (2.11) and the fact that  $X$  is contractible.

In particular, because  $N$  is indecomposable, it has no quotient supported at  $s$ . Therefore, the quotient  $M(X) \twoheadrightarrow K'$  is the maximal quotient of  $M(X)$  supported at  $s$ . That is,  $K' = H^0 i_* i^* M(X)$ , where  $i : \{s\} \rightarrow X$  is the embedding. We recall the structure of this  $K'$  in the next section.

**5.3. The maximal quotient of  $M(X)$ .** Recall from [ES12a, Corollary 5.9] the following formula for the maximal quotient  $K'$  of  $M(X)$  supported at  $s$ :

$$(5.3) \quad K' := H^0 i_* i^* M(X) \cong \delta_s^{\mu_s}.$$

The reason was simple: first,  $K'$  identifies with  $\text{Hom}(M(X), \delta_s)^* \otimes \delta_s \cong (\hat{\mathcal{O}}_{X,s})_{H(\hat{X}_s)} \otimes \delta_s$  (the second isomorphism is [ES12a, Lemma 5.10]), and then the latter was computed in [ES12a, §5.2] using [Gre75, Proposition 5.7.(iii)]. Here,  $\hat{X}_s = \text{Spf } \hat{\mathcal{O}}_{X,s}$  is the formal neighborhood of  $s$  in  $X$ .

**5.4. The family  $M(\mathcal{X})$  of  $\mathcal{D}$ -modules.** Let  $\mathcal{X} \subseteq Y$  be the total space of the family  $X_t$  for  $t \in D$ . Let  $i_t : X_t \hookrightarrow \mathcal{X}$  and  $i_{\mathcal{X}} : \mathcal{X} \hookrightarrow Y$  be the closed embeddings. We now consider the family  $(i_t)_* M(X_t)$  over the disc  $D \subseteq \mathbf{C}$  of  $\mathcal{D}$ -modules on  $\mathcal{X}$ , which can be identified with  $M(\mathcal{X})$ . (Note that, if we apply  $(i_{\mathcal{X}})_*$ , we obtain an honest  $\mathcal{O}_D - \mathcal{D}_Y$ -bimodule,  $(i_{\mathcal{X}})_* M(\mathcal{X})$ , since  $Y$  is smooth; we could work with this if desired.) Since  $X_t$  is smooth for  $t \neq 0$ , we have by [ES12a, Example 2.37] that  $M(X_t) \cong (i_t)_* \Omega_{X_t}$  for  $t \neq 0$ . Our goal is to show that  $M(\mathcal{X})$  is flat over  $D$ . Notice that, since  $D \subseteq \mathbf{C}$  is one-dimensional (complex),  $M(\mathcal{X})$  is flat if and only if it is torsion-free (i.e.,  $(i_{\mathcal{X}})_* M(\mathcal{X})$  is torsion-free, or the global sections of  $M(\mathcal{X})$  form a torsion-free  $\mathcal{O}_D$ -module). So Theorem 2.14 reduces to:

**Proposition 5.4.**  $M(\mathcal{X})$  is torsion-free over  $D$ .

In the remainder of the section we prove this result. As a first step, let  $M(\mathcal{X})_{\text{tor}} \subseteq M(\mathcal{X})$  denote the  $\mathcal{O}_D$ -torsion. Since  $M(X_t)$  is simple for  $t \neq 0$ , this torsion must be at  $t = 0$ . Moreover since  $M(\mathcal{X} \setminus \{s\})$  is torsion-free (as  $M(X \setminus \{s\}) = M(X^{\text{smth}}) = \Omega_{X^{\text{smth}}}$ ), we can conclude that  $M(\mathcal{X})_{\text{tor}}$  is supported at  $s$ , i.e., it is a direct sum of copies of  $\delta_s$  as a  $\mathcal{D}$ -module on  $\mathcal{X}$ .

Let  $M(\mathcal{X})' := M(\mathcal{X})/M(\mathcal{X})_{\text{tor}}$ . Then this is torsion-free over  $D$ , and hence a flat family of  $\mathcal{D}_Y$ -modules. Let  $M(\mathcal{X})'_t$  denote the fiber at  $t$ ; this is  $M(\mathcal{X})'_t \cong (i_t)_* \Omega_{X_t}$  for  $t \neq 0$ . By the preceding paragraph, we have an exact sequence

$$(5.5) \quad 0 \rightarrow \delta_s^r \rightarrow (i_0)_* M(X) \rightarrow M(\mathcal{X})'_0 \rightarrow 0,$$

for some  $r \geq 0$ . The proposition then reduces to showing that  $r = 0$ . To do so, we will consider  $\pi_* M(X_t)$ .

**5.5. Euler-Poincaré characteristic of  $\pi_* M(X_t)$ .** Given a finite-dimensional  $\mathbf{Z}$ -graded vector space  $V_\bullet$ , let  $\chi(V_\bullet) := \sum_{m \in \mathbf{Z}} (-1)^m \dim V_m$  denote its Euler-Poincaré characteristic.

Since  $M(\mathcal{X})'$  is a flat family,  $\chi(\pi_* M(\mathcal{X})'_t)$  is constant in  $t$ . Since  $X_t$  is smooth and homotopic to a bouquet of  $\mu_s$  spheres for  $t \neq 0$ , we conclude that

$$(5.6) \quad \chi(\pi_* M(\mathcal{X})'_0) = \chi(\pi_* M(X_t)) = \mu_s + (-1)^n, \quad t \neq 0.$$

By (5.5), we can rewrite this as

$$(5.7) \quad \chi(\pi_* M(X)) = \mu_s + (-1)^n + r.$$

The main step is to compute  $\chi(\pi_* M(X))$  using the structure of  $M(X)$  from §5.2. First of all, we evidently have

$$(5.8) \quad \chi(\pi_* M(X)) = \chi(\pi_* \mathrm{IC}(X)) + \mu_s + h^{n-1}(X \setminus \{s\}) - q,$$

where  $q \geq 0$  is the dimension of the kernel of the morphism  $N \rightarrow M(X)$ . The work now reduces to computing  $\chi(\pi_* \mathrm{IC}(X))$ .

**5.6. Computation of  $\chi(\pi_* \mathrm{IC}(X))$ .** The goal of this subsection is to prove

**Proposition 5.9.**  $\chi(\pi_* \mathrm{IC}(X)) = (-1)^n - h^{n-1}(X^{\mathrm{smth}})$ .

*Proof.* We break this into steps:

- (1) Let  $\bar{X}$  be the one-point compactification of  $X$ . Up to choosing  $X = U_s$  a small enough neighborhood of  $s$ , we have a homeomorphism  $X^{\mathrm{smth}} = U_s \setminus \{s\} \cong (0, 1) \times L$  for  $L$  (the link of the singularity) a manifold of real dimension  $2 \dim_{\mathbf{C}} X - 1$ . Thus  $\bar{X} \cong X \sqcup_{X^{\mathrm{smth}}} X$  as a topological space. Moreover,  $\bar{X}$  is homotopic to the suspension of  $X^{\mathrm{smth}}$ .
- (2) We use the classical formula of [GM80, §6.1] (cf. also, e.g., [Dur95, (1)], which is stated for algebraic varieties but extends to the analytic case): If  $\bar{X}$  is a compact analytic variety of dimension  $n$  with isolated singularities, with smooth locus  $\bar{X}^{\mathrm{smth}}$ :

$$(5.10) \quad \mathrm{IH}_i(\bar{X}) := \pi_{\dim X - i} \mathrm{IC}(\bar{X}) = \begin{cases} H_i(\bar{X}), & \text{if } i > n, \\ \mathrm{Im}(H_n(\bar{X}^{\mathrm{smth}}) \rightarrow H_n(\bar{X})), & \text{if } i = n, \\ H_i(\bar{X}^{\mathrm{smth}}), & \text{if } i < n. \end{cases}$$

- (3) We apply the Mayer-Vietoris sequence to  $\bar{X} = X \sqcup_{X^{\mathrm{smth}}} X$  to compute  $\pi_* \mathrm{IC}(\bar{X})$  in terms of  $\pi_* \mathrm{IC}(X)$  and  $\pi_* \mathrm{IC}(X^{\mathrm{smth}}) = H_{n-*}(X^{\mathrm{smth}})$ :

$$(5.11) \quad \chi(\pi_* \mathrm{IC}(\bar{X})) = 2\chi(\pi_* \mathrm{IC}(X)) - (-1)^n \chi(X^{\mathrm{smth}}),$$

where  $\chi(X^{\mathrm{smth}}) = \chi(H_*(X^{\mathrm{smth}}))$  is the Euler characteristic of  $X^{\mathrm{smth}}$ .

- (4) Now apply (5.10) to the LHS of (5.11). Note that  $H_n(\bar{X}^{\mathrm{smth}}) \rightarrow H_n(\bar{X})$  is zero since it factors through  $H_n(X)$ , which is zero as  $n = \dim_{\mathbf{C}} X > 0$  and  $X$  is contractible. We conclude that

$$(5.12) \quad \sum_{i=0}^{n-1} (-1)^{n-i} h^i(X^{\mathrm{smth}}) + \sum_{i=n+1}^{2n} (-1)^i h^i(\bar{X}) = 2\chi(\pi_* \mathrm{IC}(X)) - \sum_{i=0}^{2n} (-1)^{n-i} h^i(X^{\mathrm{smth}}).$$

- (5) Using that  $\bar{X}$  is homotopic to the suspension of  $X^{\mathrm{smth}}$  (hence  $H_i(\bar{X}) = H_{i-1}(X^{\mathrm{smth}})$  for  $i \geq 2$ ), and that  $H_{2n}(X^{\mathrm{smth}}) = 0$  as  $X^{\mathrm{smth}}$  is a noncompact real  $2n$ -manifold, the above simplifies to

$$(5.13) \quad \chi(\pi_* \mathrm{IC}(X)) = \sum_{i=0}^{n-1} (-1)^{n-i} h^i(X^{\mathrm{smth}}).$$



- (6) Finally, we apply the fact that, for  $X$  an analytically locally complete intersection of dimension  $n$  with an isolated singularity, the link of the singularity is  $(n-2)$ -connected (see [Mil68] and [Ham71, Korollar 1.3]). Therefore  $H_i(X^{\text{smth}}) = 0$  for  $1 \leq i \leq n-2$ . We obtain the proposition.  $\square$

**5.7. Proof of Theorems 2.13 and 2.14.** Putting together Proposition 5.9 and (5.8), we obtain

$$(5.14) \quad \chi(\pi_* M(X)) = (-1)^n + \mu_s - q.$$

On the other hand, comparing this with (5.7) and the fact that  $q, r \geq 0$ , we obtain

$$(5.15) \quad q = r = 0.$$

Since  $r = 0$ , this completes the proof of Proposition 5.4, as remarked there, and hence also Theorem 2.14, as also pointed out there. Since  $q = 0$ , by the definition of  $q$  in (5.8) and the comment after (5.1), Theorem 2.13 is proved as well.

## 6. PROOF OF THEOREM 3.2 AND PROPOSITION 3.4 FOR CONES OVER SMOOTH CURVES IN $\mathbf{P}^2$

For concreteness, we first prove these results in the case that  $X \subseteq \mathbf{C}^3$  is the cone over a smooth curve in  $\mathbf{P}^2$  (with vertex  $s = 0$ ), even though the general proof is essentially the same (and for most of it, we will copy and adapt the proof given here). We assume that the Poisson bivector  $\Xi_X$  has weight  $d$ , and hence that  $X$  is the zero locus of a homogeneous polynomial in  $\mathbf{C}^3$  of weight  $d+3$ .

We prove Proposition 3.4 first, and use it in the proof of Theorem 3.2.

**6.1. Proof of Proposition 3.4.** As recalled in §5.3, the maximal quotient of  $M(X)$  supported at 0 is canonically identified with  $(\hat{\mathcal{O}}_{X,0})_{H(\hat{X}_0)}$ , which is just  $\text{HP}_0(\mathcal{O}_X)$  since  $\mathcal{O}_X$  is positively graded. Now, Hamiltonian vector fields all have weight at least  $d+1$ . As a result,  $\text{HP}_0(\mathcal{O}_X) = (\mathcal{O}_X)_m$  for all  $m \leq d$ .

**6.2. Proof of Theorem 3.2.** For a contradiction, suppose that there were a direct sum decomposition

$$(6.1) \quad M(X)_d = M(X)'_d \oplus \delta_0.$$

This would induce a decomposition on solutions valued in every  $\mathcal{D}$ -module  $N$  on  $X$ ,

$$(6.2) \quad \text{Hom}(M(X)_d, N) \cong \text{Hom}(M(X)'_d, N) \oplus \text{Hom}(\delta_0, N).$$

Let  $N$  be the space of smooth, compactly supported distributions on  $X$ , i.e., smooth, compactly supported distributions on  $\mathbf{C}^3$  scheme-theoretically supported on  $X$ . Let  $w : \delta_0 \rightarrow N$  be the canonical inclusion of the delta function  $\mathcal{D}$ -module, i.e.,  $w(1) \in N$  is the delta function distribution, where  $1 \in \delta_0$  is the canonical generator. Let  $\text{Eu}$  denote the holomorphic Euler vector field on  $X$ . We will prove the following result. Let  $1 \in M(X)$  be the canonical generator. Note that  $\text{Eu}$  acts as an endomorphism of  $M(X)$ ,  $1 \cdot \Phi \mapsto 1 \cdot \text{Eu} \cdot \Phi$  for all  $\Phi \in \mathcal{D}_X$ , so let  $T_{\text{Eu}} : M(X) \rightarrow M(X)$  denote this endomorphism.

**Lemma 6.3.** There is a map  $\Phi : \text{Hom}(M(X)_d, \delta_0) \rightarrow \text{Hom}(M(X)_d, N)$  such that  $\Phi(v) \circ (T_{\text{Eu}} - d) = w \circ v$  for all  $v \in \text{Hom}(M(X)_d, \delta_0)$ .

We prove the lemma in §6.3 below. Using the lemma, we conclude the theorem as follows. First, note that  $\text{ad}(T_{\text{Eu}})$  acts semisimply on  $\text{End}(M(X))$ , since  $\text{ad}(\text{Eu})$  acts semisimply on global sections of  $\mathcal{D}_X$ , and  $\text{End}(M(X))$  is a homogeneous subquotient thereof: in more detail,

$$\text{End}(M(X)) = (\Gamma(X, H(X) \cdot \mathcal{D}_X) \setminus \Gamma(X, \mathcal{D}_X))^{H(X)}.$$

Moreover,  $\text{ad}(T_{\text{Eu}})$  acts with eigenvalue  $m - k$  on  $\text{Hom}(M(X)_k, M(X)_m)$  for all  $m, k$ , and hence it acts there by  $(m - k) \cdot \text{Id}$ . In particular,  $T_{\text{Eu}}$  is central in  $\text{End}(M(X)_m)$  for all  $m$ . Thus,  $T_{\text{Eu}}$  preserves all direct summands of  $M(X)_m$  for all  $m$ .

Suppose now that  $M(X)_d$  had a nonzero direct summand,  $K$ , supported at the origin. Then,  $T_{\text{Eu}}$  preserves  $K$ . Since  $\text{Eu}$  acts semisimply on global sections of any  $\mathcal{D}$ -module supported at the origin, it follows that  $T_{\text{Eu}}$  is a semisimple endomorphism of  $K$ . Since  $T_{\text{Eu}} - d$  is a nilpotent endomorphism of  $M(X)_d$ , it follows that  $T_{\text{Eu}} - d$  restricts to the zero endomorphism of  $K$ .

Now assume that  $K \cong \delta_0$ , up to taking a further summand. Let  $v : M(X)_d \rightarrow \delta_0$  be the corresponding projection. Then Lemma 6.3 implies that  $\Phi(v) \circ (T_{\text{Eu}} - d)$  is nonzero and factors through  $v$ , and in particular,  $\Phi(v) \circ (T_{\text{Eu}} - d)$  does not vanish on the summand  $K$ . This contradicts the fact that  $T_{\text{Eu}} - d$  restricts to zero on  $K$ . This proves the theorem.

**6.3. Proof of Lemma 6.3.** Let  $\omega := \Xi_X^{-1}$  be the meromorphic symplectic (or volume) form on  $X^{\text{smth}}$  (which is holomorphic on  $X^{\text{smth}}$ ). Then  $\omega$  has weight  $-d$ .

Let  $S_t := \{x \in X \mid |x| = t\} \subseteq \mathbf{C}^3$  be the intersection of  $X$  with the (five-dimensional) sphere of radius  $t$ , and  $B_t := \{x \in X \mid |x| \leq t\}$  the corresponding closed balls. Then  $S_t$  and  $B_t$  are compact for all  $t \in \mathbf{R}_{\geq 0}$ .

For every  $Q \in (\mathcal{O}_X)_d$ , consider the partially defined functional on  $C_c^\infty(X)$ :

$$\phi_Q : \alpha \mapsto \int_X \alpha \omega \wedge \bar{Q} \bar{\omega}.$$

For all  $m \geq 0$ , let  $(C_c^\infty(X))_{>m}$  be the subspace of (smooth compactly-supported) functions all of whose derivatives up to and including order  $m$  vanish.

**Lemma 6.4.** The functional  $\phi_Q$  converges for  $\alpha \in (C_c^\infty)_{>d}$ .

*Proof.* Let  $r : \mathbf{C}^3 \rightarrow \mathbf{R}_{\geq 0}$  be the radial function  $r(z) = |z|$ . We can rewrite

$$\phi_Q(\alpha) = \int_0^\infty dt \int_{S_t} \alpha \bar{Q} (\omega \wedge \bar{\omega}) / dr.$$

Then the above integral converges absolutely, since for  $C_\alpha > 0$  such that  $|\alpha(z)| < C_\alpha |z|^{d+1}$ , and all  $t > 0$ ,

$$\left| \int_{S_t} \alpha \bar{Q} \omega \wedge \bar{\omega} / dr \right| < C_\alpha \int_{S_t} |z|^{d+1} |Q| \omega \wedge \bar{\omega} / dr = C_\alpha \int_{S_1} |z|^{d+1} |Q| \omega \wedge \bar{\omega} / dr.$$

Letting  $C$  equal the right-hand side and  $R > 0$  be such that  $\alpha$  is supported in  $B_R$ , we obtain  $|\phi_Q(\alpha)| < C \cdot R$ , which proves the absolute convergence.  $\square$

Next, extend  $\phi_Q$  arbitrarily to a functional on all of  $C_c^\infty(X)$ . Note that the difference between any two such extensions annihilates  $C_c^\infty(X)_{>d}$ , so is a linear combination of derivatives of the delta distribution at 0 of orders  $\leq d$ . We claim that  $\phi_Q$  is annihilated by every Hamiltonian vector field  $\xi$  on  $X$ , i.e.,  $\phi_Q \in \text{Hom}(M(X), N)$ . It suffices to let  $\xi$  be homogeneous, say of weight  $m$ . Since  $\Xi$  has weight  $d$ , it follows that  $m > d$ .

First, note that  $\phi_Q \cdot \xi$  is supported at the origin, since  $\omega$  is invariant under Hamiltonian flow on  $X^{\text{smth}}$ . Now,  $\epsilon := \phi_Q \cdot (\text{Eu} - d)$  is supported at the origin. Moreover, it annihilates  $C_c^\infty(X)_{>d}$ , and hence  $\epsilon$  is a sum of homogeneous distributions supported at  $s$  of weights  $\geq -d$ . Hence  $\epsilon \cdot \xi$  is a sum of distributions supported at  $s$  of weights  $\geq m - d > 0$ , and hence is zero. Thus

$$0 = \phi_Q \cdot (\text{Eu} - d) \cdot \xi = (\phi_Q \cdot \xi) \cdot (\text{Eu} + m - d).$$

Since  $m - d > 0$  and all distributions supported at  $s$  are linear combinations of distributions of nonpositive weights, it follows that  $\phi_Q \cdot \xi = 0$ , as desired.

We also saw above that  $\phi_Q \cdot (\text{Eu} - d)$  is supported at the origin and annihilates  $C_c^\infty(X)_{>d}$ . Up to our choice of  $\phi_Q$ , i.e., adding a linear combination of derivatives of the delta function

distribution at  $s$  of weights  $\geq -d$ , we can assume that  $\phi_Q \cdot (\text{Eu} - d)$  has weight  $-d$ , and hence  $\phi_Q \cdot (\text{Eu} - d)^2 = 0$ . Thus,  $\phi_Q \in \text{Hom}(M(X)_d, N)$ . Note that this uniquely determines  $\phi_Q$  up to an element of  $(\mathcal{O}_X)_d^* \cong \text{Hom}(M(X)_d, \delta)$ . By picking a basis of  $(\mathcal{O}_X)_d$ , we can extend the assignment  $Q \mapsto \phi_Q$  to a linear map  $(\mathcal{O}_X)_d \rightarrow \text{Hom}(M(X)_d, N)$ , and any two such maps differ by a linear map valued in  $\text{Hom}(M(X)_d, \delta)$ .

Consider next the Hermitian pairing on  $\mathcal{O}_X$ ,

$$(6.5) \quad \langle P, Q \rangle := \int_{S_1} P \bar{Q} \omega \wedge \bar{\omega} / dr,$$

which restricts to nondegenerate pairings  $(\mathcal{O}_X)_m \otimes (\mathcal{O}_X)_m \rightarrow \mathbf{C}$  for all  $m \geq 0$ . We obtain an antilinear isomorphism  $\iota : \text{Hom}(M(X)_d, N) = (\mathcal{O}_X)_d^* \xrightarrow{\sim} (\mathcal{O}_X)_d$ , so that  $\langle P, \iota(v) \rangle = v(P)$ . Composing this with the linear map  $Q \mapsto \phi_Q$  above, we obtain a linear map  $\Phi : \text{Hom}(M(X)_d, \delta) \rightarrow \text{Hom}(M(X)_d, N)$ . Any two such choices of  $\Phi$  differ by an element of  $\text{Hom}(M(X)_d, \delta)$ .

We now claim that for every such  $\Phi$ , Lemma 6.3 is satisfied, up to rescaling  $\Phi$ . Since  $\text{Hom}(M(X)_d, \delta)$  is annihilated by  $T_{\text{Eu}} - d$  (i.e., all solutions of  $M(X)_d$  supported at 0 are annihilated by  $\text{Eu} - d$ ), it suffices to show this for any particular  $\Phi$ . By our definition of  $\Phi$ , we need to prove that the following holds up to a constant factor:

$$(6.6) \quad \phi_Q \cdot (\text{Eu} - d) \cdot P = \langle P, Q \rangle w(1), \quad \forall P \in (\mathcal{O}_X)_d.$$

To prove this, we construct a particular  $\phi_Q$  as follows. Let  $\beta \in C_c^\infty(X)_{>d}$  be any function such that  $\beta(0) = 1$ , and assume it is supported in the unit ball  $B_1$ . Let  $\beta_q$  be the function  $\beta_q(x) := \beta(q^{-1} \cdot x)$ , which is supported in  $B_q$ . For all  $q > 0$ , consider the projection to  $C_c^\infty(X)_{>d}$  along  $((\mathcal{O}_X)_{\leq d} \otimes (\overline{\mathcal{O}_X})_{\leq d}) \cdot \beta_q$ ,

$$\text{pr}_{>d}^q : C_c^\infty(X) \rightarrow C_c^\infty(X)_{>d}.$$

Then, for all  $q$ , we extend  $\phi_Q$  to the functional

$$\phi_{Q,q} := \phi_Q \circ \text{pr}_{>d}^q.$$

Let  $\epsilon := \phi_{Q,q} \cdot (\text{Eu} - d) \in \text{Hom}(M(X)_d, \delta)$ , which does not depend on  $q$ . Lemma 6.3 follows from the following result (once we rescale  $\Phi$  by  $-2$ ):

**Lemma 6.7.** For all  $P \in (\mathcal{O}_X)_d$ ,

$$(6.8) \quad \epsilon \cdot P = -\frac{1}{2} \langle P, Q \rangle w(1).$$

*Proof.* Since  $\epsilon \cdot (\text{Eu} - d) \cdot P$  is a multiple of  $w$  for all  $P \in (\mathcal{O}_X)_d$ , it is enough to show the identity after evaluating on a single function  $H \in C_c^\infty(X)$  with  $H(0) \neq 0$ . Let  $h \in C_c^\infty(\mathbf{R})$  be a function such that  $h(0) \neq 0$ , and let  $H(x) := h(|x|^2)$  be the corresponding spherically symmetric function on  $X$ . Then,

$$\epsilon(P \cdot H) = \int_X \bar{Q} P \text{Eu} \text{pr}_{>d}^q(H) \omega \wedge \bar{\omega},$$

for all choices of  $q$ . Taking the limit as  $q \rightarrow 0$ , this becomes

$$\int_0^\infty dt \cdot t \cdot h'(t^2) \cdot \langle P, Q \rangle = -\frac{1}{2} \langle P, Q \rangle h(0) = -\frac{1}{2} \langle P, Q \rangle H(0). \quad \square$$

## 7. PROOF OF THEOREM 3.2 AND PROPOSITION 3.4 IN GENERALITY

**7.1. Proof of Proposition 3.4.** As recalled in §5.3, the maximal quotient of  $M(X)$  supported at  $s$  is canonically identified with  $(\hat{\mathcal{O}}_{X,s})_{H(\hat{X}_s)}$ . Now,  $H(\hat{X}_s)$  is obtained by contracting  $\Xi$  with differential  $(n-2)$ -forms on the formal neighborhood  $\hat{X}_s$ . Since  $\hat{X}_s$  is conical by assumption,

differential  $(n-2)$ -forms are convergent sums of homogeneous forms of positive weight. Therefore,  $H(\hat{X}_s)_m = 0$  for  $m \leq d$ . As a result,  $((\hat{\mathcal{O}}_{X,s})_{H(\hat{X}_s)})_m = (\hat{\mathcal{O}}_{X,s})_m$  for all  $m \leq d$ .

**7.2. Proof of Theorem 3.2.** We begin as in §6.2. For convenience in referring to that section, we let  $0 := s$ , i.e., consider  $s$  to be the origin of our conical variety  $X$ . Assume for a contradiction that we have a decomposition (6.1), which induces the decomposition (6.2) on solutions valued in  $\mathcal{D}$ -modules  $N$ .

As before, let  $N$  be the space of smooth, compactly supported distributions on  $X$  (since  $X$  is conical, it embeds into an affine space, and we can define this space as the smooth, compactly supported distributions on the ambient space which are scheme-theoretically supported at  $X$ , i.e., annihilate the polynomial functions vanishing on  $X$ ; this defines  $N$  independently of the embedding up to canonical isomorphism). Just as before,  $w : \delta_s \rightarrow N$  denotes the canonical inclusion of the delta function  $\mathcal{D}$ -module,  $\text{Eu}$  denotes the holomorphic Euler vector field on  $X$ , and  $1 \in M(X)$  denotes the canonical generator. Also, as before,  $\text{Eu}$  induces an endomorphism  $T_{\text{Eu}} : M(X) \rightarrow M(X)$ .

Below we will prove that Lemma 6.3 extends to this setting. Then the theorem follows from the lemma just as before.

**7.3. Proof of Lemma 6.3 in generality.** Let  $\omega := \Xi_X^{-1}$  be the meromorphic volume form on  $X^{\text{smth}}$  (which is holomorphic on  $X^{\text{smth}}$ ). Then  $\omega$  has weight  $-d$ .

Let us assume that  $X$  is embedded into an affine space  $\mathbf{A}$  with homogeneous (positive integral weight) coordinate functions. This can be done, for example, by taking sufficiently many general homogeneous functions  $x_i \in \mathcal{O}_X$ , of weights  $a_i \geq 1$ . Let  $a$  be the least common multiple of the weights  $a_i$  of the  $x_i$ . Then we may define a radial function  $r \in C^\infty(X^{\text{smth}})$  by  $r := (\sum_i |x_i|^{2a/a_i})^{1/2a}$ , which extends continuously to  $X$  via  $r(s) = 0$ , and is smooth on  $X^{\text{smth}}$ . Moreover,  $r^{2a}$  is a smooth function on  $X$ .

Let  $S_t := \{x \in X \mid r(x) = t\}$  be the corresponding spheres of radius  $t$ , and  $B_t := \{x \in X \mid r(x) \leq t\}$  the corresponding closed balls. Then  $S_t$  and  $B_t$  are compact for all  $t \in \mathbf{R}_{\geq 0}$ .

As in §6.3, we consider the partially defined functional  $\phi_Q$ , defined by the same formula. For all  $a \in \mathbf{R}_{\geq 0}$ , let  $(C_c^\infty(X))_{>a}$  be the subspace of (smooth compactly-supported) functions  $\alpha$  such that  $\lim_{r \rightarrow 0} |\alpha/r^a| = 0$ . We can restate this as follows in terms of the derivatives of  $\alpha$ . For each coordinate function  $x_i$  on  $X$ , let  $d_i$  be its weight, and assign  $\partial_i$  weight  $-d_i$ . Then  $C_\infty^c(X)_{>a} \subseteq C_\infty^c(X)$  is the subspace of smooth functions represented by smooth functions on the affine space  $\mathbf{A}$  all whose derivatives of weights  $\geq -a$  vanish at the vertex  $s$  (this includes functions for which all derivatives up to order  $a$  vanish at  $s$ ). In particular, for  $\alpha \in C_\infty^c(X)_{>a}$ , it follows that there exists  $C_\alpha > 0$  such that  $|\alpha| < C_\alpha \cdot r^{a+1}$  (and the converse holds as well).

With these definitions, Lemma 6.4 extends to this context, with the same proof.

As before, extend  $\phi_Q$  arbitrarily to a functional on all of  $C_\infty^c(X)$ . Note that the difference between any two such extensions annihilates  $C_c^\infty(X)_{>d}$ , so is a linear combination of derivatives of the delta distribution at  $s$  of weights  $\geq -d$ . It then follows exactly as before that  $\phi_Q$  is annihilated by every Hamiltonian vector field  $\xi$  on  $X$ , i.e.,  $\phi_Q \in \text{Hom}(M(X), N)$ . We also can define the linear map  $\Phi$  and the Hermitian pairing  $\langle -, - \rangle$  on  $\mathcal{O}_X$  just as before. Then, we claim that Lemma 6.7 extends to this setting.

The proof of Lemma 6.7 is the same as before, except that we have to modify the function  $H$  as follows.

Recall from above that  $a$  was the least common multiple of the weights  $a_i$  of the coordinate functions  $x_i$  which realize the embedding of  $X$  into affine space. Let  $h \in C_c^\infty(\mathbf{R})$  be a function such

that  $h(0) \neq 0$ , and let  $H \in C_c^\infty(X)$  be the function  $H = h \circ r^{2a}$ . As before,

$$\epsilon(P \cdot H) = \int_X \bar{Q} P \operatorname{Eu pr}_{>d}^q(H) \omega \wedge \bar{\omega},$$

for all choices of  $q$ . Taking the limit as  $q \rightarrow 0$ , this becomes

$$\int_0^\infty dt (at^{2a-1} h'(t^{2a})) \cdot \langle P, Q \rangle = -\frac{1}{2} \langle P, Q \rangle h(0) = -\frac{1}{2} \langle P, Q \rangle H(s).$$

This proves Lemma 6.7 in the general setting, and hence Lemma 6.3. The theorem is proved.

## REFERENCES

- [AL98] J. Alev and T. Lambre, *Comparaison de l'homologie de Hochschild et de l'homologie de Poisson pour une déformation des surfaces de Klein*, Algebra and operator theory (Tashkent, 1997) (Dordrecht), Kluwer Acad. Publ., 1998, pp. 25–38.
- [Dur95] A. H. Durfee, *Intersection homology Betti numbers*, Proc. Amer. Math. Soc. **123** (1995), no. 4, 989–993. MR 1233968 (95e:14014)
- [ES10] P. Etingof and T. Schedler, *Poisson traces and  $\mathcal{D}$ -modules on Poisson varieties*, Geom. Funct. Anal. **20** (2010), no. 4, 958–987, arXiv:0908.3868, with an appendix by I. Losev.
- [ES12a] ———, *Coinvariants of lie algebras of vector fields on algebraic varieties*, arXiv:1211.1883, 2012.
- [ES12b] ———, *Zeroth Poisson homology of symmetric powers of isolated quasihomogeneous surface singularities*, J. Reine Angew. Math. **667** (2012), 67–88, arXiv:0907.1715. MR 2929672
- [Gil64] M. C. Gilmartin, *Every analytic variety is locally contractible*, ProQuest LLC, Ann Arbor, MI, 1964, Thesis (Ph.D.)—Princeton University. MR 2614525
- [GM80] M. Goresky and R. MacPherson, *Intersection homology theory*, Topology **19** (1980), no. 2, 135–162. MR 572580 (82b:57010)
- [Gre75] G.-M. Greuel, *Der Gauss-Manin-Zusammenhang isolierter Singularitäten von vollständigen Durchschnitten*, Math. Ann. **214** (1975), 235–266. MR 0396554 (53 #417)
- [Ham71] H. Hamm, *Lokale topologische Eigenschaften komplexer Räume*, Math. Ann. **191** (1971), 235–252. MR 0286143 (44 #3357)
- [Mil68] J. Milnor, *Singular points of complex hypersurfaces*, Annals of Mathematics Studies, No. 61, Princeton University Press, Princeton, N.J., 1968. MR MR0239612 (39 #969)